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ISBN 0-315-55480-0

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THE UNIVERSITY OF ALBERTA

SEMI-OPEN SETS AND MAPPINGS IN TOPOLOGICAL SPACES

by

Raja Mohammad Latif

A THESIS SUBMITTED TO

THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA, CANADA

FALL, 1989

THE UNIVERSITY OF ALBERTA

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ABSTRACT

In 1963, N.Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He also introduced the notion of a semi-continuous function. Later on other mappings like irresolute, semi-open, semi-closed, pre-semi-open and pre-semi-closed functions were introduced in the literature. In 1968, N.Levine also introduced the definition of a D-space by calling a topological space X to be a D-space if every nonempty open set is dense in X.

In the earlier part of this thesis we attempt to characterize, in as general a setting as we can manage, the intersection of semi-open sets. In the latter part we give some new properties and characterizations of various kinds of mappings. In this part we also prove some new properties and characterizations of D-spaces.

Chapter 1 deals with the finite and countable intersections of semi-open sets and regularly closed sets in compact metric spaces. We prove that a set in a compact metric space is the intersection of two semi-open sets if and only if it is the union of an open set and a nowhere dense set. We also show that a set in a compact metric space is the intersection of finitely many regularly closed sets if and only if it is the union of a regularly closed set and a closed

nowhere dense set. We also prove that a countable intersection of semi-open sets is a closed set minus a set of the first category.

In Chapter 2, we extend the results of Chapter 1 to separable, locally separable, locally connected and locally pathwise connected metric spaces. It is shown that a closed and nowhere dense subset of a separable, locally separable (locally compact) metric space X is the intersection of two semi-open sets in X. We also prove that a closed and nowhere dense subset of a locally connected or locally pathwise connected metric space X is the intersection of two semi-open sets.

Chapter 3 deals with various kinds of mappings in topological spaces. In this chapter we give several new properties and characterizations of semi-continuous, irresolute, semi-open, pre-semi-open and pre-semi-closed functions. For example for any two topological spaces X and Y, a function $F : X \to Y$ is semi-continuous (irresolute) if and only if for each subset B of Y, the inverse of the interior (semi-interior) of B is contained in the semi-interior of the inverse of B. The relationships between mappings are also further examined.

Chapter 4 is devoted to D-spaces. We explore some new properties and characterizations of D-spaces in terms of semi-open sets in topological spaces. For example, we show that a topological space X is a D-space if and only if the semi-closure of each nonempty semi-open set in X is X.

We also examine the characterizations of images and pre-images of D-spaces.

For example the semi-continuous image of a D-space is a D-space.

ACKNOWLEDGEMENT

Iti s a great honour and pleasure for me to express my sincere gratitude to my thesis supervisor, Dr. S.W. Willard, for his advice and encouragement. Particularly appreciated was his masterful approach and his immediate attention to the various problems which arose during the process of this thesis.

I would also like to thank the other members of my supervisory committee, Dr. A.T. Lau, Dr. M.G. Murdeshwar, Dr. R.L. McKinney and Dr. B. Linsky, for the many ways which they have been helpful to me.

Finally, I am indebted to the Canadian International Development Agency and the University of Alberta for providing me the scholarship which made this preparation possible.

TABLE OF CONTENTS

Chapter		Page
I	Semi-Open and Regularly Closed Sets In Compact Metric Spaces	1
II	Semi-Open and Regularly Closed Sets in Separable, Locally Separable, Locally Connected and Locally Pathwise Connected Metric Spaces	11
	A. Semi-Open Sets and Regularly Closed Sets In Separable Metric Spaces	11
	B. Semi-Open Sets in Locally Separable Metric Spaces	26
	C. Semi-Open Sets in Locally Connected and Locally Pathwise Connected Metrix Spaces	29
III	On Characterization of Mappings	34
	A. Semi-Continuous Functions	34
	B. Irresolute Functions	40
	C. Semi-Open Functions	48
	D. Pre-Semi-Open Functions	49
	E. Pre-Semi-Closed Functions	55

Chapter . Page

IV *D*-Spaces 60

A. Characterization of *D*-Spaces 60

B. Characterization of Images and Pre-Images of *D*-Spaces 66

Bibliography 70

CHAPTER I

SEMI-OPEN AND REGULARLY CLOSED SETS IN COMPACT METRIC SPACES

In (LEVINE [1963]), N.Levine introduced the concept of a semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U such that A contains U and is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. $SO(X)$ will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set in X but clearly a semi-open set may not be an open set in X. N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space X is never semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X. We observe that the intersection of two semi-open sets in a space X may not be a semi-open set in X. Let (X,d) be a metric space. Let $A \subseteq X$, $a \in X$ and δ be a positive real number. Then we

define the following sets.

$$O_\delta(A) = \{x \in X : d(x, A) \leq \delta\},$$

$$O_\delta(a) = \{x \in X : d(x, a) < \delta\},$$

$$O_{\bar{\delta}}(a) = \{x \in X : d(x, a) \leq \delta\}.$$

In the following our first result gives an excellent characterization of the finite intersection of semi-open sets in a topological space.

1.1. THEOREM. *The intersection of a finite number of semi-open sets in a space X is the union of an open set and a nowhere dense set in X.*

PROOF: Suppose $A = S_1 \cap S_2 \cap \cdots \cap S_n$ where each S_i is semi-open in X. Then for each $i \in \{1, 2, \ldots, n\}$, there exists G_i open in X such that

$$G_i \subseteq S_i \subseteq \overline{G_i}.$$

Let $T_i = S_i - G_i$, for each $i \in \{1, 2, \ldots, n\}$. Then each $S_i = G_i \cup T_i$ and each T_i is nowhere dense in X. Now

$$\begin{aligned} A &= \bigcap_{i=1}^{\infty} (G_i \cup T_i) \\ &= [\bigcap_{i=1}^{n} G_i] \cup [\bigcup_{i=1}^{2^n-1} (\bigcap_{k=1}^{n} E_{(i,k)})] \end{aligned}$$

where for all $i \in \{1, 2, \ldots, 2^n - 1\}$, $E_{(i,k)} = G_k$ or T_k, for $k = 1, 2, \ldots, n$, and $E_{(i,k)} = T_k$ for some $k \in \{1, 2, \ldots, n\}$. We notice that $\cap_{i=1}^{n} G_i$ is open, and all other sets in the union are nowhere dense being contained in

a nowhere dense set, whence, as is well known, their union is nowhere dense. Hence now the theorem follows by induction.

Now our first aim is to prove that the converse of the above theorem is also true in a compact metric space. For this we will make use of the result that a closed nowhere dense set in a compact metric space X is contained in the frontier of two disjoint open sets in X. To establish this latter fact, we need the following result.

1.2. THEOREM. *Let (X, ρ) be a metric space. Let A be a compact nowhere dense subset of X. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $a \in A$, there exists $x_a \in X$ such that*

$$x_a \in O_\varepsilon(a) - O_\delta(A).$$

PROOF: Let $\varepsilon > 0$. Put $\gamma = \{0_{\varepsilon/2}(a) | a \in A\}$. Then clearly γ is an open covering of A. By hypothesis A is compact. So there exist $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \cup_{k=1}^{n} O_{\varepsilon/2}(a_k)$. Since A is nowhere dense in X, $\overline{X - A} = X$. Since A is a compact subset of a metric space X, so A is closed in X. Hence $\overline{X - A} = X$. Thus if $a_k \in \overline{X - A}$, for every $k \in \{1, 2, \ldots, n\}$. Then $\rho(a_k, X - A) = 0$, for every $k \in \{1, 2, \ldots, n\}$. So for each $k \in \{1, 2 \ldots, n\}$, there exists $x_k \in X - A$ such that $\rho(a_k, x_k) < \frac{\varepsilon}{2}$. Since $\{x_1, x_2, \ldots, x_n\} \cap A = \phi$ and A is closed in X. We have $\rho(x_k, A) > 0$, for every $k \in \{1, 2, \ldots, n\}$. Put $\delta = \frac{1}{2} \min\{\rho(x_k, A) : k = 1, 2, \ldots, n\}$.

Then $\delta > 0$. Now let $a \in A$. Then $a \in \cup_{k=1}^{n} O_{\varepsilon/2}(a_k)$. So there exists $k_a \in \{1, 2, \ldots, n\}$ such that $a \in O_{\varepsilon/2}(a_{k_a})$. Then $\rho(a, a_{k_a}) < \frac{\varepsilon}{2}$. Also we have $\rho(a_{k_a}, x_{k_a}) < \varepsilon/2$. Thus, by the triangle inequality, we have $\rho(a, x_{k_a}) < \varepsilon$. Also we have, $2\delta \leq \rho(x_{k_a}, A) \leq \rho(x_{k_a}, a)$. Thus $2\delta \leq \rho(a, x_{k_a}) < \varepsilon$. Hence, we have found a $\delta > 0$ such that for every $a \in A$, there exists $x_{k_a} = x_a \in X$ such that $\delta < 2\delta \leq \rho(a, x_a) < \varepsilon$. i.e., $x_a \in O_\varepsilon(a) - O_\delta(A)$.

1.3. COROLLARY. *Let (X, ρ) be a compact metric space. Let A be a nonempty closed nowhere dense subset of X. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $a \in A$, there exists x_a in $O_\varepsilon(a) - O_{\bar{\delta}}(A)$.*

Now let us prove the result needed to be used to prove the converse of Theorem 1.1 in a compact metric space.

1.4. THEOREM. *Let (X, ρ) be a compact metric space. Let A be a closed nowhere dense subset of X. Then there exist disjoint open sets G_1 and G_2 in X such that*

$$A \subseteq \bar{G}_i - G_i, \quad \text{for } i = 1, 2.$$

PROOF: By Corollary 1.3, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for each $a \in A$, there exists x_a in $O_\varepsilon(a) - O_{\bar{\delta}_\varepsilon}(A)$. Choose $\varepsilon_1 = 1$ and, once ε_n has been chosen, choose $\varepsilon_{n+1} = \min\{\frac{1}{n+1}, \delta_{\varepsilon_n}, \frac{\varepsilon_n}{2}\}$. Then $\varepsilon_n \to 0$ as $n \to \infty$ and for any $n \in N$, and for any $a \in A$, there exists

$x_{a_n} \in O_{\varepsilon_n}(a) - O_{\delta_\varepsilon}(A)$, whence $x_{a_n} \in O_{\varepsilon_n}(a) - O_{\varepsilon_{n+1}}(A)$ as $\varepsilon_{n+1} \leq \delta_{\varepsilon_n}$. Define $S_n = O_{\varepsilon_n}(A) - O_{\varepsilon_{n+1}}(A)$, for every $n \in N$. Then S_n is open in X. Now for every $a \in A$, for each $n \in N$, there exists $x_{a_n} \in O_{\varepsilon_n}(a) - O_{\varepsilon_{n+1}}(A)$. This implies that $x_{a_n} \in S_n$ and $\rho(a, x_{a_n}) < \varepsilon_n$. Let $G_1 = \bigcup_{n=0}^{\infty} S_{2n+1}$, $G_2 = \cup_{n=1}^{\infty} S_{2n}$. Take any $n \in N$ and let $x \in S_n$. Then $x \notin O_{\varepsilon_{n+1}}(A)$. So we have $\rho(x, A) > \varepsilon_{n+1}$. Since $\{\varepsilon_n : n \in N\}$ is a strictly decreasing sequence of positive real numbers, so $\rho(x, A) > \varepsilon_{n+k}$, for all $k \in N$. Thus $x \notin O_{\varepsilon_{n+k}}(A)$, for all $k \in N$. This implies that $x \notin S_{n+k}$, for all $k \in N$. Thus $S_n \cap S_{n+k} = \phi$, for all $k \in N$. Hence $\{S_n : n \in N\}$ is a pairwise disjoint sequence of open sets in X. Thus G_1 and G_2 are mutually disjoint open sets in X. Now for all $n \in N$, $A \subseteq O_{\varepsilon_{n+1}}(A) \subseteq O_{\varepsilon_n}(A)$. Thus $A \cap S_n = \phi$, for all $n \in N$. Hence, $A \cap G_i = \phi$, for $i = 1, 2$. Now we show that $A \subseteq \bar{G}_1$. Suppose $A - \bar{G}_1 \neq \phi$. Fix $a \in A - \bar{G}_1$. Then $a \notin \bar{G}_1$. So there exists $\varepsilon > 0$ such that $G_1 \cap O_\varepsilon(a) = \phi$. Hence $S_{2n+1} \cap O_\varepsilon(a) = \phi$, for all $n \in N$. Take $m \in N$ such that $\varepsilon_{2m+1} < \varepsilon$. Then $[O_{\varepsilon_{2m+1}}(a) - O_{\varepsilon_{2m+2}}(A)] \cap O_\varepsilon(a) = \phi$. So $O_{\varepsilon_{2m+1}}(a) - O_{\varepsilon_{2m+2}}(A) = \phi$. Hence, $x_{2m+1} \notin O_{\varepsilon_{2m+1}}(a) - O_{\varepsilon_{2m+2}}(A)$ which is a contradiction. Thus $A \subseteq \bar{G}_1$. Consequently, $A \subseteq \bar{G}_1 - G_1$. Similarly, $A \subseteq \bar{G}_2 - G_2$.

Now we state and prove our aimed theorem in a compact metric space.

1.5 THEOREM. *A set A in a compact metric space X is the intersection of finitely many semi-open sets if and only if A is the union of an open set and a nowhere dense set.*

PROOF: NECESSITY. Follows from Theorem 1.1.

SUFFICIENCY. Suppose $A = H \cup N$ where H is open and N is nowhere dense. Since N is nowhere dense in X, there exist two disjoint open sets G_1 and G_2 in X such that $N \subseteq \bar{G}_i - G_i$, for $i = 1,2$. Let $S_i = G_i \cup N$, for $i = 1,2$. Then S_1 and S_2 are semi-open in X. Also $S_1 \cap S_2 = N$. But then $H \cup S_1$ and $H \cup S_2$ are semi-open. Also

$$(H \cup S_1) \cap (H \cup S_2) = H \cup (S_1 \cap S_2)$$

$$= H \cup N.$$

So $\quad (H \cup S_1) \cap (H \cup S_2) = A.$

Now our next task is to derive certain properties of finite and countable intersections of regularly closed sets in compact metric spaces.

1.6. DEFINITION. *Let (X,T) be a topological space. A closed subset of X is called regularly closed in X if and only if it is the closure of its interior.*

The following lemma gives a nice representation of a closed nowhere dense set in a compact metric space X in terms of regularly closed sets in X. It will be useful to us in what follows.

1.7. LEMMA. *In a compact metric space* (X, ρ), *a closed nowhere dense set* A *can be written as* $F_1 \cap F_2$ *where* F_1 *and* F_2 *are regularly closed.*

PROOF: Let $S_1, S_2, \ldots$ be the sequence of disjoint open sets as in the proof of Theorem 1.4. Let $G_1 = \bigcup_{n=0}^{\infty} S_{4n+1}, G_2 = \bigcup_{n=0}^{\infty} S_{4n+3}$. We note that $\{S_{4n+1} : n \in N\}$ is a locally finite family of subsets of X. Thus $\bar{G}_1 = \bigcup_{n=0}^{\infty} \bar{S}_{4n+1}$. We also claim that $\text{Int } \bar{G}_1 = \bigcup_{n=0}^{\infty} \text{Int } \bar{S}_{4n+1}$. For if $x \in \text{Int } \bar{G}_1$, then $O_\varepsilon(x) \subseteq \bigcup_{n=0}^{\infty} \bar{S}_{4n+1}$ for some $\varepsilon > 0$. Since $\{\bar{S}_{4n+1} : n \in N\}$ is a pairwise disjoint sequence of closed sets. So there exists a unique $m \in N$ such that $x \in \bar{S}_{4m+1}$ and then also easily using normality of X, there exists $\varepsilon' > 0$ such that $O_{\varepsilon'}(x) \subseteq \bar{S}_{4m+1}$. Hence, $x \in \bigcup_{n=0}^{\infty} \text{Int } \bar{S}_{4n+1}$. So $\text{Int } \bar{G}_1 \subseteq \bigcup_{n=0}^{\infty} \text{Int } \bar{S}_{4n+1}$. Note the reverse inclusion is obvious. Now we also note that $\{\text{Int } \bar{S}_{4n+1} : n \in N\}$ is a locally finite family of sets. Hence $\overline{\text{Int } \bar{G}_1} = \bigcup_{n=0}^{\infty} \overline{\text{Int } \bar{S}_{4n+1}}$. But we note that $\bar{S}_{4n+1}$ is a regularly closed set. Hence, $\overline{\text{Int } \bar{G}_1} = \bigcup_{n=0}^{\infty} \bar{S}_{4n+1} = \bar{G}_1$. Thus $\bar{G}_1$ is regularly closed. Similarly, $\bar{G}_2$ is regularly closed. Now as in Theorem 1.4, we note that $A \subseteq \bar{G}_1$ and $A \subseteq \bar{G}_2$. Also if $x \notin A$, then $\rho(x, A) > 0$. It can be seen that x is a closure point of at most two consecutive members of the sequence $\{S_n : n \in N\}$. Say $x \in \bar{S}_k$ and $x \in \bar{S}_{k+1}$. Thus $x \notin \bar{G}_1 = \bigcup_{n=0}^{\infty} \bar{S}_{4n+1}$ or $x \notin \bar{G}_2 = \bigcup_{n=0}^{\infty} \bar{S}_{4n+3}$. So $x \notin \bar{G}_1 \cap \bar{G}_2$. Conseqently, $A = \bar{G}_1 \cap \bar{G}_2$.

The next Theorem gives a characterization of the finite intersection of regularly closed sets in a compact metric space.

1.8. THEOREM. *Let* (X, ρ) *be a compact metric space. Then a subset* A *of* X *is the intersection of finitely many regularly closed sets if and only if* $A = F \cup B$ *where* F *is regularly closed and* B *is closed nowhere dense in* X.

PROOF: NECESSITY. Suppose $A = K_1 \cap K_2 \cap \cdots \cap K_n$ where each K_i is regularly closed. Then each K_i is semi-open, whence by Theorem 1.1, $A = G \cup C$ where G is open and C is nowhere dense. Let $F = \bar{G}, B = \overline{C}$. Then F is regularly closed and B is closed nowhere dense. Also we note that $A = F \cup B$.

SUFFICIENCY. Suppose $A = F \cup B$ where F is regularly closed and B is closed nowhere dense. By Lemma 1.7, we can write $B = F_1 \cap F_2$ where F_1 and F_2 are regularly closed. Hence,

$$A = F \cup (F_1 \cap F_2)$$
$$= (F \cup F_1) \cap (F \cup F_2)$$

where $F \cup F_1$ and $F \cup F_2$ are regularly closed (as it is well known that the union of two regularly closed sets is a regularly closed set).

The forthcoming lemma represents a semi-open set in terms of a regularly closed set and a nowhere dense set.

1.9. LEMMA. *Every semi-open set* S *in a space* X *can be written as* $F - T$ *where* F *is regularly closed and* T *is nowhere dense in* X.

PROOF: Let G be an open set with

$$G \subseteq S \subseteq \bar{G}.$$

Then $S = \bar{G} - [\bar{G} - S]$, and $\bar{G}$ is regularly closed while $\bar{G} - S$ is contained in $\bar{G} - G$ and is thus nowhere dense.

Finally, using the last result, we give the characterization of the countable intersection of semi-open sets in terms of a closed set and a set of first category.

1.10. THEOREM. *Every countable intersection of semi-open sets in a space X, is a closed set minus a set of first category.*

PROOF: Let S_n be semi-open in X for each $n \in N$. Then by Lemma 1.9, $S_n = F_n - T_n$ where F_n is regularly closed and T_n is nowhere dense in X. Thus

$$\begin{aligned}
\bigcap_{n=1}^{\infty} &= \bigcap_{n=1}^{\infty} (F_n - T_n) \\
&= \bigcap_{n=1}^{\infty} [F_n \cap (X - T_n)] \\
&= (\bigcap_{n=1}^{\infty} F_n) \cap [\bigcap_{n=1}^{\infty} (X - T_n)] \\
&= (\bigcap_{n=1}^{\infty} F_n) \cap (X - \bigcup_{n=1}^{\infty} T_n) \\
&= \bigcap_{n=1}^{\infty} F_n - \bigcup_{n=1}^{\infty} T_n.
\end{aligned}$$

As $\bigcap_{n=1}^{\infty} F_n$ is closed, and $\bigcup_{n=1}^{\infty} T_n$ is of first category, the theorem follows.

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CHAPTER II

SEMI-OPEN AND REGULARLY CLOSED SETS IN SEPARABLE, LOCALLY SEPARABLE, LOCALLY CONNECTED AND LOCALLY PATHWISE CONNECTED METRIC SPACES

In this chapter, we extend the results of the previous chapter to separable, locally separable, locally connected and locally pathwise connected metric spaces.

A. SEMI-OPEN SETS AND REGULARLY CLOSED SETS IN SEPARABLE METRIC SPACES

In this main section of this chapter, we give a characterization of the intersection of semi-open sets and regularly closed sets in separable metric spaces.

Our first result shows that a nowhere dense set in a separable metric space can be represented in terms of two semi-open sets.

2.1. THEOREM. *If* X *is a separable metric space and* N *is a nowhere dense subset of* X, *then there exist two semi-open sets* S_1 *and* S_2 *in* X *such that* $N = S_1 \cap S_2$.

PROOF: It is well-known that by Urysohn's metrization Theorem, a separable metrizable space X can be embedded as a subspace of the Hilbert cube $I^{\aleph_0}$. So there exists $\ell : X \to I^{\aleph_0}$ an embedding. Let $K = c\ell_{I^{\aleph_0}}\ell(X)$. Then K is a compact metric space, $\ell(X)$ is dense in K and $\ell : X \to K$ is

an embedding. Now to verify that $\ell(N)$ is nowhere dense in K, let $\mathcal{V}$ be a nonempty open subset of K. Then $\ell^{-1}(\mathcal{V})$ being an open nonempty subset of K, we can fix $x \in \ell^{-1}(\mathcal{V}) \cap (X - cl_X N)$ as N is nowhere dense in X. Then $\ell(x) \in \mathcal{V}$ and $x \notin cl_X N$. But as $\ell : X \to \ell(X)$ is a homeomorphism, so $x \notin cl_X N$ implies that $\ell(x) \notin cl_{\ell(X)}\ell(N)$. This shows that $\ell(N)$ is nowhere dense in $\ell(X)$. Then it follows easily that $\ell(N)$ is nowhere dense in K as $\ell(X)$ is dense in K. Hence by Theorem 1.4, there exist disjoint open sets G_1 and G_2 in K such that

$$cl_K \ell(N) \subseteq cl_K G_i - G_i, \quad \text{for} \quad i = 1, 2.$$

Let

$$H_i = G_i \cap \ell(X), \quad \text{for} \quad i = 1, 2.$$

Then H_i is open in $\ell(X)$ for each $i = 1, 2$, and H_1 and H_2 are disjoint. Also $\ell(N) \subseteq cl_K H_i - H_i$, for $i = 1, 2$. For if $y \in \ell(N)$, then every K-neighborhood of y meets G_i whence every $\ell(X)$-neighborhood of y meets H_i, for $i = 1, 2$. Now set $T_i = H_i \cup \ell(N)$, for $i = 1, 2$. Then clearly T_1 and T_2 are semi-open in $\ell(X)$ and $\ell(N) = T_1 \cap T_2$.

Now let $S_1 = \ell^{-1}(T_1)$ and $S_2 = \ell^{-1}(T_2)$. Then as $\ell : X \to \ell(X)$ is a homeomophism, so it follows immediately that S_1 and S_2 are semi-open in X and $N = S_1 \cap S_2$.

2.2. COROLLARY. *If X is a separable metric space and $A = G \cup N$ where G is open and N is nowhere dense, then $A = S_1 \cap S_2$ where S_1 and S_2 are semi-open.*

PROOF: By Theorem 2.1, we can write $N = T_1 \cap T_2$ where T_1 and T_2 are semi-open. Then

$$G \cup N = G \cup (T_1 \cap T_2)$$
$$= (G \cup T_1) \cap (G \cup T_2).$$

Clearly $S_1 = G \cup T_1$ and $S_2 = G \cup T_2$ are semi-open and $A = S_1 \cap S_2$.

Now we will show that a closed set in a separable metric space X can be represented in terms of regularly closed sets in X. To see this we need the following lemmas.

2.3. LEMMA. *If X is dense in a space K and A is open in K, then*

$$X \cap cl_K A = cl_X(X \cap A).$$

PROOF: $X \cap cl_K A$ is closed in X and contains $X \cap A$, so $cl_X(X \cap A) \subseteq X \cap cl_K A$. On the other hand, suppose $x \in X \cap cl_K A$. Then $x \in X$ and every K-neighborhood of x meets A. Let U be an open neighborhood of x in X. Then $U = \mathcal{V} \cap X$, where $\mathcal{V}$ is open in K. Now $\mathcal{V}$ is a K-neighborhood of x and hence meets A, whence $\mathcal{V} \cap A$ is a nonempty

open set in K, so $(\mathcal{V} \cap A) \cap X \neq \phi$, and therefore $(\mathcal{V} \cap X) \cap A \neq \phi$ whence $U \cap (X \cap A) \neq \phi$. Thus $x \in cl_X(X \cap A)$.

The next lemma describes the regularly closed property possessed by the intersection of a regularly closed set and a dense subspace of a space.

2.4. LEMMA. *Let X be dense in a space K. If R is regularly closed in K, then $R \cap X$ is regularly closed in X.*

PROOF: We have $R = cl_K \operatorname{Int}_K R$. But then using lemma 2.3, we have

$$X \cap R = X \cap cl_X \operatorname{Int}_K R$$

$$= cl_X[X \cap \operatorname{Int}_K R].$$

Hence $X \cap R$ is the closure of an open set in X, whence $X \cap R$ is regularly closed.

2.5. LEMMA. *Every closed set A in a space X is the union of a regularly closed set and a nowhere dense closed set.*

PROOF: It is enough to note that

$$A = (cl_X \operatorname{Int} A) \cup (A - \operatorname{Int} A).$$

Now let us state and prove our central result.

2.6. THEOREM. *Every closed set in a separable metric space is the intersection of two regularly closed sets.*

PROOF: Let X be a separable metric space and let A be a closed subset of X. Embed X as a dense subset of a compact metric space K. Then $A = X \cap cl_K A$. By Lemma 2.5, $cl_K A = R \cup N$ where R is a regularly closed set and N is a closed nowhere dense set in K. Then by Lemma 1.7, $N = F_1 \cap F_2$ where F_1 and F_2 are regularly closed in K. Thus

$$\begin{aligned} cl_K A &= R \cup (F_1 \cap F_2) \\ &= (R \cup F_1) \cap (R \cup F_2) \\ &= T_1 \cap T_2 \end{aligned}$$

where $T_1 = R \cup F_1$ and $T_2 = R \cup F_2$. Also clearly T_1 and T_2 are regularly closed in K. Hence

$$\begin{aligned} A &= X \cap (T_1 \cap T_2) \\ &= (X \cap T_1) \cap (X \cap T_2) \\ &= E_1 \cap E_2. \end{aligned}$$

By Lemma 2.4, $E_1 = X \cap T_1$ and $E_2 = X \cap T_2$ are regularly closed in X.

Now our next result is that in a separable metric space X, the difference of a closed set and a set of the first category can be characterized in terms of a countable intersection of semi-open sets in X.

For this purpose, we will make use of the following lemma.

2.7. LEMMA. *If* R *is regularly closed and* T *is nowhere dense in a space* X, *then* $R-T$ *is semi-open.*

PROOF: Let $G = \text{Int}(R-T)$. Then $G \subseteq (R-T)$. We claim that $(R-T) \subseteq c\ell\,\text{Int}(R-T) = c\ell\, G$. To justify our claim, let $x \in (R-T)$ and let $\mathcal{V}$ be an open set containing x. It is sufficient to show that $\mathcal{V} \cap \text{Int}(R-T) \neq \phi$. Now $R = c\ell\,\text{Int}\,R$ and $x \in R$, so $\mathcal{V}$ meets $\text{Int}\,R$. Hence, since T is nowhere dense, $\mathcal{V}$ meets $(\text{Int}\,R - c\ell\,T)$. But $(\text{Int}\,R - c\ell\,T)$ is open and is contained in $(R-T)$, so $(\text{Int}\,R - c\ell\,T) \subseteq \text{Int}(R-T)$. Hence $\mathcal{V}$ meets $\text{Int}(R-T)$ as required.

Now let us state and prove the result we have been aiming for.

2.8. THEOREM. *Let* X *be a separable metric space. Let* $A = F-T$ *where* F *is closed and* T *is first category. Then* $A = \bigcap_{n=1}^{\infty} S_n$ *where each* S_n *is semi-open.*

PROOF: By Theorem 2.6, we can write

$$F = R_1 \cap R_2$$

where R_1 and R_2 are regularly closed. Since T is of first category, we

can write $T = \bigcap_{n=1}^{\infty} T_n$ where each T_n is nowhere dense. Then

$$\begin{aligned} A &= F - T \\ &= (R_1 \cap R_2) - (\bigcup_{n=1}^{\infty} T_n) \\ &= \bigcap_{n=1}^{\infty} [(R_1 \cap R_2) - T_n] \\ &= \bigcap_{n=1}^{\infty} [(R_1 - T_n) \cap (R_2 - T_n)] \end{aligned}$$

By Lemma 2.7, for all $n \in N$, $R_1 - T_n$ and $R_2 - T_n$ are semi-open.

The following proposition shows that for each given infinite cardinal number τ, there exists a compact T_2-space Y containing a dense discrete subspace X of cardinalily τ such that each non-isolated point in Y is not the intersection of two semi-open sets in Y.

2.9. PROPOSITION. *Let X be an infinite discrete space. Let βX be the Stone-Čech compactification of X. Let $p \in \beta X - X$. Then $\{p\}$ is nowhere dense but $\{p\}$ is not the intersection of two semi-open sets in βX.*

PROOF: We note that $\{p\}$ is closed in βX and p is non-isolated because βX is compact T_2 and X is dense in βX. Thus $\{p\}$ is nowhere dense in βX.

In order to prove that $\{p\}$ is not the intersection of two semi-open sets, it is enough to show that for each pair of disjoint open sets, say G and H in βX, $cl_{\beta X} G \cap cl_{\beta X} H = \phi$. For this let $U = G \cap X$ and $V = H \cap X$. Then since X is dense in βX, so we have $cl_{\beta X} U = cl_{\beta X} G$

and $cl_{\beta X}\mathcal{V} = cl_{\beta X}H$. But then U and $\mathcal{V}$ are disjoint zero-sets in X, so $cl_{\beta X}U \cap cl_{\beta X}\mathcal{V} = \phi$. Hence $cl_{\beta X}G \cap cl_{\beta X}H = \phi$ as required.

In particular the preceeding proposition shows that a non-isolated point in a separable compact T_2-space may not be an intersection of two semi-open sets.

Now we will show that a non-isolated point in a T_4-space with a discrete subspace may not be an intersection of countably many semi-open sets.

Let X be a Tychonoff space. Then the set $\{t \in \beta X : X$ is c-embedded in $X \cup \{t\}\}$ denoted by υX (Pronounced: upsilon X) is called the real compactification of X. We call X realcompact if $\upsilon X = X$.

We call a cardinal m measureable if a set X of cardinal m admits a $\{0,1\}$-valued measure μ such that $\mu(X) = 1$, and $\mu(\{x\}) = 0$ for every $x \in X$.

2.10. THEOREM. *A discrete space is realcompact if and only if its cardinal is nonmeasurable.*

PROOF: 12.2 of (GILLMAN AND JERISON [1960])

2.11. PROPOSITION. *Let X be a discrete space with its cardinal measurable. By Theorem 2.10, fix $p \in \mathcal{V}X - X$. Then $\{p\}$ is not an intersection of countably many semi-open sets in υX.*

PROOF: Contrariwise if possible, then suppose that there exists a sequence $\{S_n : n \in N\}$ of semi-open sets such that $\{p\} = \bigcap_{n=1}^{\infty} S_n$. Then there exists a sequence $\{G_n : n \in N\}$ of open sets such that $G_n \subseteq S_n \subseteq c\ell_{vX} G_n$, for each $n \in N$. This implies that $\bigcap_{n=1}^{\infty} G_n \subseteq \bigcap_{n=1}^{\infty} S_n = \{p\} \subseteq \bigcap_{n=1}^{\infty} c\ell_{vX} G_n$. Since p being a point in $vX - X$, is not a G_δ in vX. Thus $\bigcap_{n=1}^{\infty} G_n = \phi$.

Now for each $n \in N$, put $H_n = G_n \cap X$. Since X is dense in vX and each G_n is open in vX. So it follows that

$$c\ell_{vX} H_n = c\ell_{vX} G_n, \quad \text{for each } n \in N.$$

Then since each H_n is a zero-set in X and $\bigcap_{n=1}^{\infty} H_n = \phi$, so it implies that $\bigcap_{n=1}^{\infty} c\ell_{vX} H_n = \phi$. Hence $\bigcap_{n=1}^{\infty} c\ell_{vX} G_n = \phi$. But then ~~$\{p\} = \phi$~~, which is impossible. Hence we conclude that $\{p\}$ is not the intersection of countably many semi-open sets in vX.

Our next aim is to give the characterization of a separable closed and nowhere dense subspace of a metrizable space in terms of semi-open sets. To achieve our aim, we will need the following well-known results. Their proofs may be seen in some standard book on General Topology. Firstly let us recall totally bounded spaces. A subset D of a metric space X with the metric ρ will be called an ε-net in X if for every point $x \in X$ there exists an $x' \in D$ such that $\rho(x, x') < \varepsilon$. A metric space (X, ρ) is said to be totally bounded if for each $\varepsilon > 0$ there exists a finite ε-net $\{x_1, x_2, \ldots, x_k\}$ in X. A metrizable space X will be called metrizable in a totally bounded

manner provided that there exists a metric ρ in X such that (X, ρ) is totally bounded.

2.12. THEOREM. *In order that a metrizable space X be metrizable in a totally bounded manner it is necessary and sufficient that X be separable.*

2.13. THEOREM. *If X is any metrizable space, A is a closed subset of X, and ρ is a compatible metric on A, then ρ can be extended to a compatible metric on X.*

2.14. THEOREM. *Every metric space X can be isometrically embedded as a dense subset of a complete space.*

2.15. THEOREM. *A set A is nowhere dense in the space X if and only if every nonempty open set contains a nonempty open set which is disjoint from A.*

We also need the following Lemma.

2.16. LEMMA. *Let (X^*, ρ^*) be a complete metric space. Let $A \subseteq X^*$ such that A is totally bounded in X^*. Then $cl_{X^*} A = \bar{A}$ is compact.*

PROOF: Let $\varepsilon > 0$. Since A is totally bounded.

There exists a finite subset D of A such that for each $a \in A$, there exists $d_a \in D$ such that $\rho^*(a, d_a) < \varepsilon/2$. Let $x \in \bar{A}$. Then $A \cap O_{\varepsilon/2}(x) \neq \phi$. Fix any $a_x \in A \cap O_{\varepsilon/2}(x)$.

Then $a_x \in A$ and $a_x \in O_{\varepsilon/2}(x)$. i.e., $a_x \in A$ and $\rho^*(a_x, x) < \frac{\varepsilon}{2}$. But also for $a_x \in A$, we have $d_{a_x} \in D$ such that $\rho^*(a_x, d_{a_x}) < \varepsilon/2$. Thus $\rho^*(x, d_{a_x}) \leq \rho^*(a, a_x) + \rho^*(a_x, d_{a_x}) < \varepsilon$. Hence for each $x \in \bar{A}$, there exists $d_x^* = d_{a_x} \in D$ such that $\rho^*(x, d_x^*) < \varepsilon$ and D is a finite subset of $\bar{A}$. Hence $\bar{A}$ is totally bounded in X^*.

Now also $\bar{A}$ being a closed subspace of a complete space $X^*, \bar{A}$ is complete. Thus $\bar{A}$ being complete and totally bounded, $\bar{A}$ is compact.

Now we are ready to prove our Theorem.

2.17. THEOREM. *Let X be a metrizable space. Let A be a separable, closed and nowhere dense subset of X. Then there exist two semi-open sets R_1 and R_2 in X such that $A = R_1 \cap R_2$.*

PROOF: By using Theorem 2.12, let ρ be a compatible metric on A such that (A, ρ) is totally bounded. Then by applying Theorem 2.13, we can extend ρ to a compatible metric on X. Now by making use of Theorem 2.14, there exists a compatible metric space (X^*, ρ^*) and $f : X \to X^*$ such that f is an isometry of X onto $f(X)$ and $f(X)$ is dense in X^*.

Now to see that $f(A)$ is totally bounded in X^*, let $\varepsilon > 0$. Since A is totally bounded, there exists a finite subset D of A such that for each $a \in A$, there exists $d_a \in D$ such that $\rho(a, d_a) < \varepsilon$. So for any $a \in A$, $f(d_a) \in f(D)$ and $\rho^*(f(a), f(d_a)) = \rho(a, d_a) < \varepsilon$. Also $f(D)$ is a finite subset of $f(A)$. Thus $f(A)$ is totally bounded in X^*.

Put $Z = cl_{X^*} f(A)$. Then by Lemma 2.16, Z is compact.

Now to show that $f(A)$ is nowhere dense in $f(X)$, let G be a nonempty open subset of $f(X)$. Then clearly $f^{-1}(G)$ is a nonempty open subset of X. Since A is nowhere dense in X, by Theorem 2.15 $f(A)$ is nowhere dense in $f(X)$.

Now to verify that $f(A)$ is nowhere dense in X^*, let G^* be a nonempty open subset of X^*. Then as $f(X)$ is dense in X^*. $\tilde{G} = G^* \cap f(X)$ is a nonempty open subset of $f(X)$. Hence by Theorem 2.15, there exists a nonempty open subset $\tilde{H}$ of $f(X)$ such that $\tilde{H} \subseteq \tilde{G}$ and $\tilde{H} \cap f(A) = \phi$.

Let H^* be a nonempty open subset of X^* such that $\tilde{H} = H^* \cap f(X)$ and $H^* \cap f(A) = \phi$. Then $K^* = H^* \cap G^*$ is a nonempty open subset of X^* such that $K^* \subseteq G^*$ and $K^* \cap f(A) = \phi$. Hence by Theorem 2.15, $f(A)$ is nowhere dense in X^*.

Thus Z is a closed compact nowhere dense subset of X^*. So $Z = S_1 \cap S_2$ where S_1 and S_2 are semi-open sets in X^*. Let U_1 and U_2 be open sets in X^* such that

$$U_i \subseteq S_i \subseteq cl_{X^*} U_i, \text{for} \quad i = 1,2.$$

Put $W_i = f(X) \cap U_i$, $T_i = f(X) \cap S_i$, for $i = 1,2$. Then since $f(X)$ is dense in X^*, so

$$cl_{X^*} W_i = cl_{X^*} U_i, \quad \text{for} \quad i = 1,2.$$

This implies that

$$cl_{f(X)}W_i = cl_{f(X)}U_i, \quad \text{for} \quad i = 1, 2.$$

Now

$$U_i \cap f(X) \subseteq S_i \cap f(X) \subseteq (cl_{X^*}U_i) \cap f(X) = cl_{f(X)}U_i, \quad \text{for} \quad i = 1, 2.$$

Thus we have

$$W_i \subseteq T_i \subseteq cl_{f(X)}W_i, \quad \text{for} \quad i = 1, 2.$$

Hence T_1 and T_2 are semi-open in $f(X)$. Also

$$\begin{aligned} cl_{f(X)}f(A) &= f(X) \cap cl_{X^*}f(A) \\ &= f(X) \cap Z \\ &= f(X) \cap (S_1 \cap S_2) \\ &= (f(X) \cap S_1) \cap (f(X) \cap S_2) \\ &= T_1 \cap T_2. \end{aligned}$$

Since $f(A)$ is closed in $f(X)$. So $f(A) = T_1 \cap T_2$. Let $R_1 = f^{-1}(T_1)$, $R_2 = f^{-1}(T_2)$. Since $f : X \to f(X)$ is a homeomorphism, we have $A = R_1 \cap R_2$ and also it follows easily that R_1 and R_2 are semi-open in X.

2.18. THEOREM. *Let (X, ρ) be a metric space. Let A be a closed separable subset of X. Then there exist $T_1, T_2 \in SO(X)$ such that $A = T_1 \cap T_2$.*

PROOF: Note that $A = \text{Int } A \cup (A - \text{Int } A)$. Clearly $(A - \text{Int } A)$ is a separable, closed and nowhere dense subset of X. So by Theorem 2.17, there exist $S_1, S_2 \in SO(X)$ such that

$$A - \text{Int } A = S_1 \cap S_2.$$

Hence

$$\begin{aligned} A &= (\text{Int } A) \cup (S_1 \cap S_2) \\ &= (\text{Int } (A) \cup S_1) \cap (\text{Int } (A) \cup S_2). \end{aligned}$$

Let $T_i = \text{Int}(A) \cup S_i$, for $i = 1, 2$. Then $A = T_1 \cap T_2$. Also T_1 and T_2 are semi-open sets each being a union of an open set and a semi-open set.

2.19. DEFINITION: A set A in a topological space X is said to be semi-closed if and only if there exists a closed set F such that

$$\text{Int } F \subseteq A \subseteq F.$$

2.20. THEOREM. *A set A in a space X is semi-closed if and only if Int $\bar{A} \subseteq A$.*

PROOF: This follows easily.

Now we give a characterization of a semi-closed set in a separable metric space.

2.21. THEOREM. *Let* (X, ρ) *be a separable metric space. Let* A *be a semi-closed subset of* X. *Then there exist two semi-open sets* S_1 *and* S_2 *such that* $A = S_1 \cap S_2$.

PROOF: By Theorem 2.20, $\text{Int } \bar{A} \subseteq A$. Hence $A = (\text{Int } \bar{A}) \cup (A - \text{Int } \bar{A})$. Take $B = A - (\text{Int } \bar{A})$. Then clearly $\text{Int } \overline{B} \subseteq \text{Int } \bar{A}$. Since $B \cap (\text{Int } \bar{A}) = \phi$ So $B \cap (\text{Int } \overline{B}) = \phi$. This implies that $\overline{B} \cap (\text{Int } \overline{B}) = \phi$. Hence $\text{Int } \overline{B} = \phi$. Thus B is nowhere dense in X. Now by Theorem 2.1, there exist $T_1, T_2, \in SO(X)$ such that

$$B = A - \text{Int } \bar{A} = T_1 \cap T_2.$$

It follows that

$$\begin{aligned} A &= (\text{Int } \bar{A}) \cup (T_1 \cap T_2) \\ &= (\text{Int}(\bar{A}) \cup T_1) \cap (\text{Int}(\bar{A}) \cup T_2). \end{aligned}$$

Let $S_i = \text{Int}(\bar{A}) \cup T_i$, for $i = 1, 2$. Then S_1 and S_2 are semi-open. Also $A = S_1 \cap S_2$.

B. SEMI-OPEN SETS IN LOCALLY SEPARABLE METRIC SPACES

In this section, we extend some results of the previous section to locally separable metric spaces.

2.22. THEOREM. *For a metric space to be locally separable it is necessary and sufficient that it be a disjoint union of open separable subsets.*

PROOF: Theorem of (SIERPINSKI [1933]).

2.23. THEOREM. *Let* (X,ρ) *be a locally separable metric space. Let* A *be a nowhere dense subset of* X. *Then there exist* $S,T \in SO(X)$ *such that* $A = S \cap T$.

PROOF: If X is separable, then we already know that $A = S \cap T$ for some $S,T \in SO(X)$. If X is non-separable, then by Theorem 2.22, there exists $\{X_\alpha : \alpha \in I\}$ a pairwise disjoint class of separable open sets such that $X = \cup_{\alpha\in I} X_\alpha$. For each $\alpha \in I$, let $A_\alpha = A \cap X_\alpha$. Then each A_α is nowhere dense in X_α. So by Theorem 2.1, there exist S_α, $T_\alpha \in SO(X_\alpha)$ such that $A_\alpha = S_\alpha \cap T_\alpha$. Then for $\alpha \in I$, there exist open sets G_α and H_α in X_α such that $G_\alpha \subseteq S_\alpha \subseteq cl_{X_\alpha} G_\alpha$ and $H_\alpha \subseteq T_\alpha \subseteq cl_{X_\alpha} H_\alpha$. Clearly G_α and H_α are open in X, for all $\alpha \in I$. We claim that $cl_{X_\alpha} G_\alpha \subseteq cl_X G_\alpha$, for each $\alpha \in I$. For that let $x \in cl_{X_\alpha} G_\alpha$. Let U be any open subset of X such that $x \in U$. Then $x \in U \cap X_\alpha$ and $U \cap X_\alpha$

is open in X_α. So $(U \cap X_\alpha) \cap G_\alpha \neq \phi$. This implies that $U \cap G_\alpha \neq \phi$. Hence $x \in cl_X G_\alpha$. Thus $cl_{X_\alpha} G_\alpha \subseteq cl_X G_\alpha$. Similarly,

$$cl_{X_\alpha} H_\alpha \subseteq cl_X H_\alpha.$$

We conclude that S_α and T_α are semi-open in X, for all $\alpha \in I$. Put $S = \cup_{\alpha \in I} S_\alpha$, $T = \cup_{\alpha \in I} T_\alpha$. Then clearly S and T are semi-open in X. Further, $S_\alpha \cap T_\beta = \phi$, for all $\alpha, \beta \in I$ such that $\alpha \neq \beta$. Hence $S_\alpha \cap T = S_\alpha \cap T_\alpha = A_\alpha$, for each $\alpha \in I$. Thus

$$\cup_{\alpha \in I}(S_\alpha \cap T) = \cup_{\alpha \in I} A_\alpha.$$

So

$$(\cup_{\alpha \in I} S_\alpha) \cap T = \cup_{\alpha \in I}(A \cap X_\alpha).$$

Therefore

$$S \cap T = A \cap (\cup_{\alpha \in I} X_\alpha) = A \cap X = A,$$

i.e., $A = S \cap T$.

2.24. COROLLARY. *Let (X, ρ) be a locally compact metric space. Let A be a nowhere dense subset of X. Then there exist $S, T \in SO(X)$ such that $A = S \cap T$.*

PROOF: Note that every locally compact metric space is locally separable. The desired result now follows from Theorem 2.23.

Our next Theorem characterizes the union of an open set and a nowhere dense set in terms of two semi-open sets in a locally separable metric space.

2.25 THEOREM. *Let* (X,ρ) *be a locally separable metric space. Let* $A \subseteq X$. *Then* A *is the intersection of finitely many semi-open sets in* X *if and only if* A *is the union of an open set and a nowhere dense set in* X.

PROOF: $\Longrightarrow$: Follows from Theorem 1.1.

$\Longleftarrow$: Let $A = G \cup N$ where G is an open set in X and N is a nowhere dense set in X. By Theorem 2.23, there exist two semi-open sets S_1 and S_2 in X such that $N = S_1 \cap S_2$. Thus

$$A = G \cup (S_1 \cap S_2)$$

$$= (G \cup S_1) \cap (G \cup S_2).$$

Let

$$S = G \cup S_1,\ T = G \cup S_2.$$

Then S and T are semi-open in X. Also $A = S \cap T$.

2.26. COROLLARY. *Let* (X,ρ) *be a locally compact metric space. Let* $A \subseteq X$. *Then* A *is the intersection of finitely many semi-open sets in* X *if and only if* A *is the union of an open set and a nowhere dense set in* X.

PROOF: We note that every locally compact metric space is locally separable. Now the result follows immediately from Theorem 2.25.

C. SEMI-OPEN SETS IN LOCALLY CONNECTED AND LOCALLY PATHWISE CONNECTED METRIC SPACES

In this section of the chapter, we characterize the intersection of semi-open sets in locally connected and locally pathwise connected metric spaces.

The following Theorem deals with a locally connected metric space.

2.27. THEOREM. *Let* (X,ρ) *be a locally connected metric space. Let* A *be a nonempty closed nowhere dense subset of* X. *Then* A *is the intersection of two semi-open sets in* X.

PROOF: Define $f: X \to [0,\infty)$ by

$$f(x) = \rho(x,A), \quad \text{for all} \quad x \in X.$$

Then f is continuous and $A = f^{-1}(\{0\})$. For each $n \in N$, let

$$G_n = (\frac{1}{2n+1}, \frac{1}{2n}), \quad H_n = (\frac{1}{2n}, \frac{1}{2n-1}).$$

Let $G = \bigcap_{n=1}^{\infty} G_n$, $H = \bigcap_{n=1}^{\infty} H_n$. Then G and H are disjoint open sets in $[0,\infty)$ and $0 \in \bar{G}$, $0 \in \overline{H}$. So $S = G \cup \{0\}$ and $T = H \cup \{0\}$ are semi-open sets in $[0,\infty)$ with $S \cap T = \{0\}$. Let $S^* = f^{-1}(S)$, $T^* = f^{-1}(T)$. Then

$$A = f^{-1}(\{0\}) = f^{-1}(S) \cap f^{-1}(T) = S^* \cap T^*.$$

Now we show that S^* is semi-open. Clearly $f^{-1}(G)$ is open and

$$f^{-1}(G) \subseteq f^{-1}(S) = S^* = f^{-1}(G) \cup f^{-1}(\{0\}) = f^{-1}(G) \cup A.$$

So it is enough to show that $A \subseteq \overline{f^{-1}(G)}$. Let $x \in A$. Since X is locally connected, let U be any open connected neighborhood of x. Then $f(x) = 0 \in f(U)$ and $f(U)$ is connected. Since A is nowhere dense and closed, we can fix $y \in U \cap (X - A)$ and hence $f(y) > 0$ and $f(y) \in f(U)$. While $f(U)$ being connected, is an open interval, so $[0, f(y)] \subseteq f(U)$. Let $n^* \in N$ such that $\frac{1}{2n^*} < f(y)$. Then

$$[\frac{1}{2n^*+1}, \frac{1}{2n^*}] \subseteq f(U).$$

Let

$$z^* = \frac{1}{2}(\frac{1}{2n^*} - \frac{1}{2n^*+1}) \in (\frac{1}{2n^*+1}, \frac{1}{2n^*}) = G_{n^*}.$$

Then $z^* \in f(U)$. So there exists $x^* \in U$ such that $f(x^*) = z^*$ But then $f(x^*) \in G_{n^*} \subseteq G$ implies that $x^* \in f^{-1}(G)$. Hence,

$$x^* \in U \cap f^{-1}(G) \neq \phi.$$

So $x \in \overline{f^{-1}(G)}$. Thus $A \subseteq \overline{f^{-1}(G)}$ as required. Similarly, T^* is semi-open in X^*. Thus A is the intersection of two semi-open sets S^* and T^* in X.

2.28. COROLLARY. *Let* (X, ρ) *be a locally arcwise connected metric space. Let* A *be a closed and nowhere dense subset of* X. *Then* A *is the intersection of two semi-open sets in* X.

PROOF: It is enough to note that a locally arcwise connected metric space is locally connected and then the result follows from Theorem 2.27.

Our next result is concerned with a locally pathwise connected metric space.

2.29. THEOREM. *Let* $(X.\rho)$ *be a locally pathwise connected metric space. Let* A *be a closed and nowhere dense subset of* X. *Then there exist semi-open sets* S^* *and* T^* *in* X *such that*

$$A = S^* \cap T^*.$$

PROOF: Define $f : X \to [0, \infty)$ by

$$f(x) = \rho(x, A), \quad \text{for all} \quad x \in X.$$

Then f is continuous and $A = f^{-1}(\{0\})$. Now for each $n \in N$, let $G_n = (\frac{1}{2n+1}, \frac{1}{2n})$, $H_n = (\frac{1}{2n}, \frac{1}{2n-1})$. Also put

$$G = \bigcap_{n=1}^{\infty}, \quad H = \bigcap_{n=1}^{\infty} H_n.$$

Then G and H are disjoint open sets in $[0,\infty)$ and $0 \in \bar{G}$, $0 \in \overline{H}$. Thus $S = G \cup \{0\}$ and $T = H \cup \{0\}$ are semi-open sets in $[0,\infty)$ with $S \cap T = \{0\}$. Let $S^* = f^{-1}(S)$ and $T^* = f^{-1}(T)$. Then

$$A = f^{-1}(\{0\}) = f^{-1}(S) \cap f^{-1}(T) = S^* \cap T^*.$$

Next we show that S^* is semi-open. Clearly $f^{-1}(G)$ is open as f is continuous. Also

$$f^{-1}(G) \subseteq f^{-1}(S) = S^* = f^{-1}(G) \cup f^{-1}(\{0\}) = f^{-1}(G) \cup A.$$

So it suffices to verify that $A \subseteq \overline{f^{-1}(G)}$. Let $x \in A$. By hypothesis X is locally pathwise connected. So let U be any (basic) pathwise connected neighborhood of x. Then as A is closed nowhere dense, so we can fix $y \in U \cap (X - A)$. Then $x \neq y$; $x, y \in U$ and U is pathwise connected. So there exists a continuous function $g : [0,1] = I \to X$ such that

$$g(I) \subseteq U, \quad g(0) = x, \quad g(1) = y.$$

Let $Y = g(I)$, $h = f \circ g$. Then h is continuous and $h(I) = f(g(I)) = f(Y)$. Since $h(I)$ is connected and so it is an interval. Also $h(0) = f(g(0)) = f(x) = 0 \in h(I)$, $h(1) = f(g(1)) = f(y) \in h(I)$, $f(y) > 0$ as $y \notin A$. Thus we have $[0, f(y)] = h(I)$. Hence there exists $n^* \in N$ such that

$$G_{n^*} = (\frac{1}{2n^*+1}, \frac{1}{2n^*}) \subseteq [0, f(y)].$$

Fix any $z^* \in G_{n^*}$. Then $z^* \in G$. Also $z^* \in h(I) = f(Y)$ implies that there exists $y^* \in Y$ such that $f(y^*) = z^*$. Hence $y^* \in U$ as $Y \subseteq U$. Also $y^* \in f^{-1}(G)$ as $z^* = f(y^*) \in G$. Thus $y^* \in U \cap f^{-1}(G)$ implies that $U \cap f^{-1}(G) \neq \phi$. Thus $x \in \overline{f^{-1}(G)}$. So it follows that $A \subseteq \overline{f^{-1}(G)}$ as required. Similarly, one can prove that T^* is also semi-open. Also it is already noted that $A = S^* \cap T^*$.

CHAPTER III

ON CHARACTERIZATION OF MAPPINGS

In the literature various kinds of mappings between topological spaces have been defined. Their properties and characterizations also have been studied. The purpose of the present chapter is to continue to explore further properties and characterizations of semi-continuous, irresolute, semi-open, pre-semi-open and pre-semi-closed functions.

A. SEMI-CONTINUOUS FUNCTIONS

In (LEVINE [1963]), N. Levine introduced the notion of semi-continuous mapping. In (BISWAS [1970]), N. Biswas gave a certain characterization of semi-continuous mappings. The purpose of this section is to investigate further properties and characterizations of such semi-continuous functions.

3.1. DEFINITION: Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then $f : X \to Y$ is called semi-continuous if and only if, for any V open in Y, $f^{-1}(V) \in SO(X)$.

3.2. DEFINITION: Let X be a topological space. We say that a set $M_x \subseteq X$ is a semi-neighbourhood of a point $x \in X$ if and only if there exists a semi-open set A such that $x \in A \subseteq M_x$.

3.3. DEFINITION: Let X be a topological space. Let $p \in X$ and $A \subseteq X$. We say that p is a semi-limit point of A if and only if $U \cap (A - \{p\}) \neq \phi$,

for every $U \in SO(X)$ such that $p \in U$. The set of all semi-limit points of A is said to be the semi-derived set of A and is denoted by A^{sd}.

$A \cup A^{sd}$ is defined to be the semi-closure of A and is denoted by $scl\ A$.

3.4. THEOREM. *Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then the following are equivalent.*

(a) The function f is semi-continuous.

(b) For each point p in X and each open set O in Y with $f(p) \in O$, there is a semi-open set A in X such that $p \in A$ and $f(A) \subseteq O$.

(c) For each x in X and each neighbourhood U of $f(x)$, there is a semi-neighbourhood V of x such that $f(V) \subseteq U$.

(d) For each subset B of Y, $scl[f^{-1}(B)] \subseteq f^{-1}(cl\ B)$.

PROOF: Theorem 4 of (BISWAS [1970]).

3.5. THEOREM. *Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then the function f is semi-continuous if and only if $f(A^{sd}) \subseteq f(A) \cup [f(A)]^d$, for every $A \subseteq X$.*

PROOF: NECESSITY. Let $A \subseteq X$ and $a_0 \in A^{sd}$. Assume that $f(a_0) \notin f(A)$. Let V be a neighbourhood of $f(a_0)$. Since f is semi-continuous, so by Theorem 3.4, there exists a semi-neighbourhood U of a_0 for which

$f(U) \subseteq \mathcal{V}$. From $a_o \in A^{sd}$, it implies that $U \cap A \neq \phi$. Fix $a \in U \cap A$. Then $a \in U$ and $a \in A$. Hence $f(a) \in \mathcal{V}$ and $f(a) \in f(A)$. Since $f(a_0) \notin f(A)$. So $f(a) \neq f(a_0)$. Thus every neighbourhood of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$. Hence we conclude that $f(a) \in [f(A)]^d$. This proves necessity.

SUFFICIENCY. Assume that f is not semi-continuous. Then by Theorem 3.2, there exists $a_0 \in X$ and a neighbourhood $\mathcal{V}$ of $f(a_0)$ such that every semi-neighbourhood U of a_0 contains at least one element $a \in U$ such that $f(a) \notin \mathcal{V}$. Put $A = \{a \in X : f(a) \notin \mathcal{V}\}$. Then $a_0 \notin A$ since $f(a_0) \in \mathcal{V}$, and therefore $f(a_0) \notin f(A)$; also $f(a_0) \notin [f(A)]^d$ since

$$f(A) \cap (\mathcal{V} - \{f(a_0)\}) = \phi.$$

Thus $f(A^{sd})$ is not contained in $f(A) \cup [f(A)]^d$. But this is a contradiction to our hypothesis. Hence, f is semi-continuous.

3.6. THEOREM. *Let X and Y be topological spaces. Let $f : X \to Y$ be an injection. Then f is semi-continuous if and only if*

$$f(A^{sd}) \subseteq [f(A)]^d, \quad \textit{for every} \quad A \subseteq X.$$

PROOF: NECESSITY. Let $A \subseteq X, a_0 \in A^{sd}$ and $\mathcal{V}$ be a neighbourhood of $f(a_0)$. Since f is semi-continuous, by Theorem 3.4, there exists a semi-neighbourhood U of a_0 for which $f(U) \subseteq \mathcal{V}$. But $a_0 \in A^{sd}$, hence

there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and, since f is an injection, $f(a) \neq f(a_0)$. Thus every neighbourhood V of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in [f(A)]^d$. We have therefore, $f(A^{sd}) \subseteq [f(A)]^d$.

SUFFICIENCY. Follows from Theorem 3.5.

3.7. THEOREM. *Let* (X,T) *be a topological space. Let* $A \subseteq X$. *Then*

$$Int\ A = X - c\ell(X - A), \quad for\ all \quad A \subseteq X.$$

PROOF: This is easy.

3.8. DEFINITION: Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a semi-interior point of A if and only if there exists $U \in SO(X)$ such that $x \in U \subseteq A$. The set of all semi-interior points of A is said to be the semi-interior of A and is denoted be s Int A.

3.9. THEOREM. *Let* (X,T) *be a topological space. Let* $B \subseteq X$. *Then*

$$s\ Int\ B = X - sc\ell(X - B).$$

PROOF: Clearly, s Int $B \subseteq B$ Thus $X - B \subseteq X - s$ Int B, from which it follows that $sc\ell(X - B) \subseteq sc\ell(X - s$ Int $B)$, i.e., $sc\ell(X - B) \subseteq X - s$ Int B. Hence s Int $B \subseteq X - sc\ell(X - B)$. On the other hand if $x \in X - sc\ell(X - B)$,

then $x \notin scl(x-B)$. Hence there exists $U_x \in SO(X)$ such that $x \in U_x$ and $U_x \cap (X-B) = \phi$. Then $x \in U_x \in SO(X)$ and $U_x \subseteq B$ so that $x \in s$ Int B. This shows that $X - scl(X-B) \subseteq s$ Int B Thus, s Int $B = X - scl(X-B)$.

3.10. THEOREM. *Let X and Y be topological spaces. Then a function $f: X \to Y$ is semi-continuous if and only*

$$f^{-1}(Int\ B) \subseteq s\ Int\ f^{-1}(B), \quad \text{for each } B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Then by Theorem 3.7,

$$\text{Int } B = Y - cl(Y-B).$$

Thus

$$\begin{aligned} f^{-1}(\text{Int } B) &= f^{-1}(Y - cl(Y-B)) \\ &= X - f^{-1}cl(Y-B). \end{aligned}$$

Since f is semi-continuous, we have by Theorem 3.4,

$$scl\,f^{-1}(Y-B) \subseteq f^{-1}(cl(Y-B)).$$

Hence, $f^{-1}(\text{Int } B) \subseteq X - scl\, f^{-1}(Y-B)$. Thus $f^{-1}(\text{Int } B) \subseteq X - scl(X - f^{-1}(B))$. Now by applying Theorem 3.9, we conclude that

$$f^{-1}(\text{Int } B) \subseteq s \text{ Int } f^{-1}(B).$$

SUFFICIENCY. Let B be an arbitrary open set in Y. Then $B = \text{Int } B$. Hence, by hypothesis,

$$f^{-1}(B) \subseteq s \text{ Int } f^{-1}(B).$$

But $s \text{ Int } f^{-1}(B) \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B) = s \text{ Int } f^{-1}(B)$. Thus the set $f^{-1}(B)$ is semi-open. Consequently f is semi-continuous.

3.11. THEOREM. *Let X and Y be topological spaces. Let $f : X \to Y$ be a mapping and $g : X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for every point $x \in X$. Then g is semi-continuous if and only if f is semi-continuous.*

PROOF: NECESSITY. Let $x \in X$ and $\mathcal{V}$ be any open set containing $f(x)$. Then $X \times \mathcal{V}$ is an open set in $X \times Y$ containing $g(x)$. Since g is semi-continuous, there exists a semi-open set U containing x such that $g(U) \subseteq X \times \mathcal{V}$. Since g is the graph mapping of f, we have $f(U) \subseteq \mathcal{V}$. By Theorem 3.4, it follows that f is semi-continuous.

SUFFICIENCY. Suppose f is semi-continuous. Let $W = U \times \mathcal{V}$ be a basic open set in $X \times Y$. Then $g^{-1}(W) = U \cap f^{-1}(\mathcal{V})$. Then U is open in X and $f^{-1}(\mathcal{V})$ is semi-open in X. Hence $g^{-1}(W)$ being the intersection of an open set and a semi-open set, is semi-open. Since the union of a collection of semi-open sets is semi-open. Thus g is semi-continuous.

B. IRRESOLUTE FUNCTIONS

In this section, the function to be considered are those for which inverses of semi-open sets are semi-open. We investigate some new properties and characterizations of such functions.

3.12. DEFINITION: Let X and Y be topological spaces. A function $f: X \to Y$ is said to be irresolute if and only if, for any semi-open subset S of Y, $f^{-1}(S)$ is semi-open in X.

3.13. THEOREM. *A function $f: X \to Y$, where X and Y are topological spaces, is irresolute if and only if for each point p in X and each semi-open set B in Y with $f(p) \in B$, there is a semi-open set A in X such that $p \in A$, $f(A) \subseteq B$.*

PROOF: NECESSITY. Let $p \in X$ and $B \in SO(Y)$ such that $f(p) \in B$. Let $A = f^{-1}(B)$. Since f is irresolute, A is semi-open in X. Also $p \in f^{-1}(B) = A$ as $f(p) \in B$. We also have

$$f(A) = ff^{-1}(B) \subseteq B.$$

SUFFICIENCY. Let $B \in SO(Y)$, let $A = f^{-1}(B)$. We show that A is semi-open in X. For this let $x \in A$. It implies that $f(x) \in B$. Then by hypothesis, there exists $A_x \in SO(X)$ such that $x \in A_x$ and $f(A_x) \subseteq B$. Then $A_x \subseteq f^{-1}f(A_x) \subseteq f^{-1}(B) = A$. Thus $A = \cup_{x \in A} A_x$. It follows that A is semi-open in X. Hence f is irresolute.

3.14. THEOREM. *A function* $f : X \to Y$ *is irresolute if and only if for each* x *in* X, *the inverse of every semi-neighbourhood of* $f(x)$ *is a semi-neighbourhood of* x.

PROOF: NECESSITY. Let $x \in X$ and let B be a semi-neighbourhood of $f(x)$. Then there exists $U \in SO(X)$ such that $f(x) \in U \subseteq B$. This implies that $x \in f^{-1}(U) \subseteq f^{-1}(B)$. Since f is irresolute, so $f^{-1}(U) \in SO(X)$. Hence $f^{-1}(B)$ is a semi-neighbourhood of x.

SUFFICIENCY. Let $B \in SO(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. But then B, being semi-open, is a semi-neighbourhood of $f(x)$. So by hypothesis, $A = f^{-1}(B)$ is a semi-neighbourhood of x. Hence by definition, there exists $A_x \in SO(X)$ such that $x \in A_x \subseteq A$. Thus

$$A = \bigcup_{x \in A} A_x.$$

It follows that A is semi-open in X. Therefore, f is irresolute.

3.15. THEOREM. *A function* $f : X \to Y$ *is irresolute if and only if for each* x *in* X *and each semi-neighbourhood* U *of* $f(x)$, *there is a semi-neighbourhood* $\mathcal{V}$ *of* x *such that* $f(\mathcal{V}) \subseteq U$.

PROOF: NECESSITY. Let $x \in X$ and let U be a semi-neighbourhood of $f(x)$. Then there exists $O_{f(x)} \in SO(Y)$ such that $f(x) \in O_{f(x)} \subseteq U$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(U)$. By hypothesis, $f^{-1}(O_{f(x)}) \in SO(X)$.

Let $V = f^{-1}(U)$. Then it follows that V is a semi-neighbourhood of x and $f(V) = ff^{-1}(U) \subseteq U$.

SUFFICIENCY. Let $B \in SO(Y)$. Put $O = f^{-1}(B)$. Let $x \in O$. Then $f(x) \in B$. Thus B is a semi-neighbourhood of $f(x)$. So by hypothesis, there exists a semi-neighbourhood V_x of x such that $f(V_x) \subseteq B$. Thus it follows that $x \in V_x \subseteq f^{-1}f(V_x) \subseteq f^{-1}(B) = O$. Since V_x is a semi-neighbourhood of x, so there exists an $O_x \in SO(X)$ such that $x \in O_x \subseteq V_x$. Hence $x \in O_x \subseteq O, O_x \in SO(X)$. Thus $O = \cup_{x \in O} O_x$. It follows that O is semi-open in X. Therefore, f is irresolute.

3.16. THEOREM. *A function $f : X \to Y$ is irresolute, if and only if, for all $B \subseteq Y$, $scl[f^{-1}(B)] \subseteq f^{-1}[scl(B)]$.*

PROOF: Theorem 1.6 of [CROSSLEY AND HILDEBRAND: 1972]

3.17. THEOREM. *Let $f : X \to Y$ be a function. Then f is irresolute if and only if $f^{-1}(s\ Int\ B) \subseteq s\ Int\ f^{-1}(B)$, for all $B \subseteq Y$.*

PROOF: NECESSITY. Let $B \subseteq Y$. Then by Theorem 3.9,

$$s\ \mathrm{Int}\ B = Y - scl(Y - B).$$

Hence

$$f^{-1}(s\text{ Int } B) = f^{-1}(Y - scl(Y - B))$$
$$= f^{-1}(Y) - f^{-1}(scl(Y - B))$$
$$= X - f^{-1}(scl(Y - B))$$

Since f is irresolute, so by Theorem 3.16,

$$scl\ f^{-1}(Y - B) \subseteq f^{-1}[scl(Y - B)].$$

Hence $f^{-1}(s\text{ Int } B) \subseteq X - scl\ f^{-1}(Y - B)]$. Thus $f^{-1}(s\text{ Int } B) \subseteq X - scl[f^{-1}(Y) - f^{-1}(B)]$. Therefore $f^{-1}(s\text{ Int } B) \subseteq X - scl[X - f^{-1}(B)]$. By Theorem 3.9, it follows that

$$f^{-1}(s\text{ Int } B) \subseteq s\text{ Int } f^{-1}(B).$$

SUFFICIENCY. Let $B \in SO(Y)$. Then clearly $B = s\text{ Int } B$. So by hypothesis, $f^{-1}(B) \subseteq s\text{ Int } f^{-1}(B)$. But we know that $s\text{ Int } f^{-1}(B) \subseteq f^{-1}(B)$. Thus $f^{-1}(B) = s\text{ Int } f^{-1}(B)$. Hence $f^{-1}(B) \in SO(X)$. Therefore, f is irresolute.

3.18. THEOREM. *A function $f : X \to Y$ is irresolute if and only if*

$$f(A^{sd}) \subseteq f(A) \cup (f(A))^{sd}, \quad \text{for all } A \subseteq X.$$

PROOF: NECESSITY. Let $f : X \to Y$, be an irresolute.

Let $A \subseteq X$, and $a_0 \in A^{sd}$. Assume that $f(r_0) \notin f(A)$ and let $\mathcal{V}$ denote a semi-neighbourhood of $f(a_0)$. Since f is irresolute, so by Theorem 3.15, there exists a semi-neighbourhood U of a_0 such that $f(U) \subseteq \mathcal{V}$. From $a_0 \in A^{sd}$, it follows that $U \cap A \neq \phi$; there exists, therefore, at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in \mathcal{V}$. Since $f(a_0) \notin f(A)$, we have $f(a) \neq f(a_0)$. Thus every semi-neighbourhood of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$, consequently, $f(a_0) \in (f(A))^{sd}$. This proves necessity of the condition.

SUFFICIENCY. Assume that f is not irresolute. Then by Theorem 3.15, there exists $a_0 \in X$ and a semi-neighbourhood $\mathcal{V}$ of $f(a_0)$ such that every semi-neighbourhood U of a_0 contains at least one element $a \in U$ for which $f(a) \notin \mathcal{V}$. Put $A = \{a \in X : f(a) \notin \mathcal{V}\}$. Then $a_0 \notin A$ since $f(a_0) \in \mathcal{V}$, and therefore $f(a_0) \notin f(A)$; also $f(a_0) \notin [f(A)]^{sd}$ since $f(A) \cap (\mathcal{V} - \{f(a_0)\}) = \phi$. It follows that $f(a_0) \in f(A^{sd}) - [f(A) \cup (f(A))^{sd}] \neq \phi$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.

3.19. THEOREM. *Let $f : X \to X^*$ be a 1-1 function. Then f is an irresolute if and only if*

$$f(A^{sd}) \subseteq (f(A))^{sd}, \quad \textit{forall} \quad A \subseteq X.$$

PROOF: NECESSITY. Let f be irresolute. Let $A \subseteq X$, $a_0 \in A^{sd}$ and $\mathcal{V}$ be a semi-neighbourhood of $f(a_0)$. Since f is irresolute, so by Theorem

3.15, there exists a semi-neighbourhood. U of a_0 such that $f(U) \subseteq \mathcal{V}$. But $a_0 \in A^{sd}$; hence there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and, since f is 1-1, $f(a) \neq f(a_0)$. Thus every semi-neighbourhood $\mathcal{V}$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in (f(A))^{sd}$. We have therefore $f(A^{sd}) \subseteq (f(A))^{sd}$.

SUFFICIENCY: Follows from Theorem 3.18.

3.20. THEOREM. *Let X and Y be topological spaces and $Z = X \times Y$ be the topological product. Let $A \in SO(X)$ and $B \in SO(Y)$. Then $A \times B \in SO(X \times Y)$.*

PROOF: Theorem 11 of (LEVINE [1963]).

3.21. THEOREM. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. If g is irresolute, then f is irresolute.*

PROOF: Let $x \in X$. and $\mathcal{V} \in SO(Y)$ such that $f(x) \in \mathcal{V}$. Then by Theorem 3.20, $X \times \mathcal{V}$ is a semi-open subset of $X \times Y$ containing $g(x)$ and hence by Theorem 3.13, there exists $U \in SO(X)$ such that $x \in U$ and $g(U) \subseteq X \times \mathcal{V}$. By the definition of g, we have $f(U) \subseteq \mathcal{V}$. Therefore, by Theorem 3.13, f is irresolute.

The converse of Theorem 3.21 is not true as the following example shows.

3.22. EXAMPLE. *Let* $X = \{1,2,3\}$ *and*

$$T = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}.$$

Let $f : (X,T) \to (X,T)$ *be a function defined as follows:*

$$f(1) = 2, \quad f(2) = 1 \quad \text{and} \quad f(3) = 3.$$

Then f *is irresolute but*

$$g : (X,T) \to (X \times X, T \times T)$$

is not irresolute. Let $\mathcal{V} = \{(1,1),(1,3),(3,3)\}$ *Then clearly* $\mathcal{V}$ *is semi-open in* $X \times X$ *but* $g^{-1}(\mathcal{V}) = \{3\}$ *is not semi-open in* X. *This shows that* g *is not irresolute.*

2.23. DEFINITION: A topological space X is said to be semi-T_2 if for each two distinct points $x, y \in X$, there exist $U, \mathcal{V} \in SO(X)$ such that $x \in U$, $y \in \mathcal{V}$ and $U \cap \mathcal{V} = \phi$.

We note that every T_2-space is a semi-T_2-space but a semi-T_2-space need not be a T_2-space. For example, let $X = \{1,2,3\}$ and

$$T = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}$$

be a topology on X. Then (X,T) is a semi-T_2-space but it is not a T_2-space.

3.24. THEOREM. *If* $f : X \to Y$ *is an irresolute injection and* Y *is semi-*T_2*, then* X *is semi-*T_2.

PROOF: Let x_1 and x_2 be two distinct points of X. Since f is injective and Y is semi-T_2, there exist $\mathcal{V}_1, \mathcal{V}_2 \in SO(Y)$ such that $f(x_1) \in \mathcal{V}_1$, $f(x_2) \in \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$. Then $x_1 \in f^{-1}(\mathcal{V}_1)$, $x_2 \in f^{-1}(\mathcal{V}_2)$ and $f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) = \phi$. Since f is irresolute, so $f^{-1}(\mathcal{V}_1), f^{-1}(\mathcal{V}_2) \in SO(X)$. Thus X is semi-T_2.

3.25. THEOREM. *If* $f : X \to Y$ *is irresolute and* Y *is semi-*T_2*, then the set* $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ *is semi-closed in* $X \times X$.

PROOF: Let $(x_1, x_2) \in (X \times X) - A$. Then $f(x_1) \neq f(x_2)$. Since Y is semi-T_2, there exist $\mathcal{V}_1, \mathcal{V}_2 \in SO(X)$ such that $f(x_1) \in \mathcal{V}_1$, $f(x_2) \in \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$. Then $x_1 \in f^{-1}(\mathcal{V}_1)$, $x_2 \in f^{-1}(\mathcal{V}_2)$ and $f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) = \phi$. Since f is irresolute, $f^{-1}(\mathcal{V}_1)$ and $f^{-1}(\mathcal{V}_2)$ are semi-open in X. Put $U = f^{-1}(\mathcal{V}_1) \times f^{-1}(\mathcal{V}_2)$. Then $(x_1, x_2) \in U \in SO(X \times X)$ by Theorem 3.20. Also clearly $U \cap A = \phi$. Therefore, $(x_1, x_2) \notin scl\, A$ and hence A is semi-closed in $X \times X$.

3.26. THEOREM. *If* $f : X \to Y$ *is irresolute and* Y *is semi-*T_2*, then the graph* $G(f)$ *is semi-closed in* $X \times Y$.

PROOF: Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is semi-T_2, there exist $\mathcal{V}, W \in SO(Y)$ such that $y \in \mathcal{V}$, $f(x) \in W$ and $\mathcal{V} \cap W = \phi$.

Since f is irresolute, so by Theorem 3.13, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subseteq W$. Then $f(U) \cap V = \phi$. Therefore, $(x,y) \in U \times V \in SO(X \times Y)$ and $(U \times V) \cap G(f) = \phi$. This shows that $(x,y) \notin scl(G(f))$ and thus $G(f)$ is semi-closed in $X \times Y$.

C. SEMI-OPEN FUNCTIONS

In (BISWAS [1969]), N. Biswas defined semi-open mappings as a generalization of open mappings and investigated several properties of such mappings. In (NOIRI [1973]), T. Noiri gave some more properties of semi-open mappings. The purpose of this section is to add another characterization of a semi-open mapping.

3.27. DEFINITION: Let X and Y be topological spaces. A function $f: X \to Y$ is called semi-open if and only if for each open set U in X, $f(U)$ is a semi-open set in Y.

3.28. THEOREM. *Let X and Y be two topological spaces. A necessary and sufficient condition for a mapping $f: X \to Y$ to be semi-open is that $f^{-1}(scl B) \subseteq cl\, f^{-1}(B)$ for every subset B of Y.*

PROOF: Theorem 2 of (NOIRI [1973]).

3.29. THEOREM. *Let* X *and* Y *be topological spaces. A function* $f : X \to Y$ *is semi-open if and only if*

$$Int[f^{-1}(B)] \subseteq f^{-1}(s\ Int\ B), \quad \text{for all} \quad B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Then Int $f^{-1}(B)$ is open in X. Since f is semi-open so $f[\text{Int } f^{-1}(B)]$ is semi-open in Y. Also we have

$$f[\text{Int } f^{-1}(B)] \subseteq ff^{-1}(B) \subseteq B.$$

Hence, $f[\text{Int } f^{-1}(B)] \subseteq s \text{ Int } B$. Thus Int $f^{-1}(B) \subseteq f^{-1}(s \text{ Int } B)$.

SUFFICIENCY. Let $B \subseteq Y$. Then $B^c = (Y - B) \subseteq Y$. By hypothesis, Int $f^{-1}(B^c) \subseteq f^{-1}[s \text{ Int}(B^c)]$. By Theorem 3.7 and Theorem 3.9, we get

$$X - c\ell[X - f^{-1}(B^c)] \subseteq f^{-1}[Y - sc\ell(Y - B^c)].$$

Hence, $X - c\ell[f^{-1}(Y) - f^{-1}(B^c)] \subseteq X - f^{-1}(sc\ell\ B)$. Thus $X - c\ell[f^{-1}(B)] \subseteq X - f^{-1}(sc\ell\ B)$. Therefore $f^{-1}(sc\ell\ B) \subseteq c\ell[f^{-1}(B)]$. Then by Theorem 3.28, it follows that f is semi-open.

D. PRE-SEMI-OPEN FUNCTIONS

The purpose of this section is to discuss some new properties and characterizations of pre-semi-open functions.

3.30. DEFINITION: Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be pre-semi-open if and only if for each $A \in SO(X)$, $f(A) \in SO(Y)$.

3.31. THEOREM. *Let $f : X \to Y$ and $g : Y \to Z$ be any two pre-semi-open functions. Then the composition function $g \circ f : X \to Z$ is a pre-semi-open function.*

PROOF: Let $U \in SO(X)$. Then $f(U) \in SO(Y)$ since f is pre-semi-open. But then $g(f(U)) \in SO(Z)$ as g is pre-semi-open. Thus $(g \circ f)(U)$ is semi-open in Z. Hence, $g \circ f$ is pre-semi-open.

3.32. THEOREM. *$f : X \to Y$ is pre-semi-open if and only if for each $x \in X$ and for every $U \in SO(X)$ such that $x \in U$, there exists $V \in SO(Y)$ such that $f(x) \in V$ and $V \subseteq f(U)$.*

PROOF: Routine.

3.33. THEOREM. *$f : X \to Y$ is pre-semi-open if and only if for each $x \in X$ and for every semi-neighbourhood U of x in X, there exists a semi-neighbourhood V of $f(x)$ in Y such that $V \subseteq f(U)$.*

PROOF: NECESSITY. Let $x \in X$ and let U be a semi-neighbourhood of x. Then there exists $W \in SO(X)$ such that $x \in W \subseteq U$. Then

$f(x) \in f(W) \subseteq f(U)$. But $f(W) \in SO(Y)$ as f is pre-semi-open. Hence $\mathcal{V} = f(W)$ is a semi-neighbourhood of $f(x)$ and $\mathcal{V} \subseteq f(U)$.

SUFFICIENCY. Let $U \in SO(X)$. Let $x \in U$. Then U is a semi-neighbourhood of x. So by hypothesis, there exists a semi-neighbourhood $\mathcal{V}_{f(x)}$ of $f(x)$ such that $f(x) \in \mathcal{V}_{f(x)} \subseteq f(U)$. It follows at once that $f(U)$ is a semi-neighbourhood of $f(x)$. Thus $f(U)$ is a semi-neighbourhood of each of its points. Thus $f(U)$ is semi-open. Hence f is pre-semi-open.

3.34. THEOREM. *Let X and Y be topological spaces. Then a function $f : X \to Y$ is pre-semi-open if and only if $f(s\ Int\ A) \subseteq s\ Int\ f(A)$, forall $A \subseteq X$.*

PROOF: NECESSITY. Let $A \subseteq X$ Let $x \in s$ Int A. Then there exists $U_x \in SO(X)$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in SO(Y)$. Hence $f(x) \in s$ Int $f(A)$. Thus

$$f(s \text{ Int } A) \subseteq s \text{ Int } f(A).$$

SUFFICIENCY. Let $U \in SO(X)$. Then by hypothesis,

$$f(s \text{ Int } U) \subseteq s \text{ Int } f(U).$$

Since s Int $U = U$ as U is semi-open. Also s Int $f(U) \subseteq f(U)$. Hence $f(U) = s$ Int $f(U)$. Thus $f(U)$ is semi open in Y. So f is pre-semi-open.

We remark that the equality does not hold in the above Theorem as the following example shows.

3.35. EXAMPLE: Let $X = Y = \{1,2\}$. Suppose X is antidiscrete and Y is discrete. Let $f = Id., A = \{1\}$. Then $\phi = f(s \text{ Int } A) \neq \text{Int } f(A) = \{1\}$.

3.36. THEOREM. *A function $f : X \to Y$ is pre-semi-open if and only if*

$$s \text{ Int } f^{-1}(B) \subseteq f^{-1}(s \text{ Int } B), \quad \text{for all } B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Since $s \text{ Int } f^{-1}(B)$ is semi-open in X and f is pre-semi-open, $f[s \text{ Int } f^{-1}(B)]$ is semi-open in Y. Also we have

$$f[s \text{ Int } f^{-1}(B)] \subseteq ff^{-1}(B) \subseteq B.$$

Hence, $f[s \text{ Int } f^{-1}(B)] \subseteq s \text{ Int } B$. Therefore $s \text{ Int } f^{-1}(B) \subseteq f^{-1}(s \text{ Int } B)$.

SUFFICIENCY. Let $A \subseteq X$. Then $f(A) \subseteq Y$.

Hence by hypothesis, we obtain

$$s \text{ Int } A \subseteq s \text{ Int } f^{-1}[f(A)] \subseteq f^{-1}[s \text{ Int } f(A)].$$

This implies that $f(s \text{ Int } A) \subseteq ff^{-1}[s \text{ Int } f(A)] \subseteq s \text{ Int } f(A)$. Thus $f(s \text{ Int } A) \subseteq s \text{ Int } f(A)$, for all $A \subseteq X$. Hence, by Theorem 3.34, f is pre-semi-open.

3.37. THEOREM. *Let X and Y be topological spaces. A necessary and sufficient condition for a mapping $f : X \to Y$ to be pre-semi-open is that $f^{-1}(scl\ B) \subseteq scl\ f^{-1}(B)$ for every subset B of Y.*

PROOF: NECESSITY. Let $B \subseteq Y$. Then $(Y - B) \subseteq Y$. By applying Theorem 3.36,

$$s\ \mathrm{Int}[f^{-1}(Y - B)] \subseteq f^{-1}[s\ \mathrm{Int}(Y - B)].$$

This implies that

$$s\ \mathrm{Int}[X - f^{-1}(B)] \subseteq f^{-1}[s\ \mathrm{Int}(Y - B)].$$

Now by applying Theorem 3.9, we obtain

$$X - scl\ f^{-1}(B) \subseteq f^{-1}(Y - scl\ B)$$

So $X - scl\ f^{-1}(B) \subseteq X - f^{-1}(scl\ B)$. Hence $f^{-1}(scl\ B) \subseteq scl\ f^{-1}(B)$.

SUFFICIENCY Let $B \subseteq Y$. Then $(Y - B) \subseteq Y$. By hypothesis, $f^{-1}[scl(Y - B)] \subseteq scl\ f^{-1}(Y - B)$. This implies $X - scl\ f^{-1}(Y - B) \subseteq X - f^{-1}[scl(Y - B)]$ Hence $X - scl[X - f^{-1}(B)] \subseteq f^{-1}[Y - scl(Y - B)]$. By applying Theorem 3.9, we have

$$s\ \mathrm{Int}\ f^{-1}(B) \subseteq f^{-1}(s\ \mathrm{Int}\ B).$$

Now from Theorem 3.36, it follows that f is pre-semi-open.

3.38. THEOREM. *Let X and Y be two topological spaces. If $f : X \to Y$ is an open and semi-continuous mapping, then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X.*

PROOF: Theorem 3 of (NOIRI [1973]).

3.39. THEOREM. *Let* X, Y *and* Z *be three topological spaces, and let* $f : X \to Y$, $g : Y \to Z$ *be two mappings so that* $g \circ f : X \to Z$ *a pre-semi-open mapping. Then*

(1) If f *is open, semi-continuous and surjective, then* g *is pre-semi-open.*

(2) If g *is open, semi-continuous and injective, then* f *is pre-semi-open.*

PROOF:

(1) Let $\mathcal{V}$ be an arbitrary semi-open set in Y. Then $f^{-1}(\mathcal{V})$ is semi-open in X because f is irresolute by Theorem 3.38. Since $g \circ f$ is pre-semi-open and f is surjective, $g(\mathcal{V}) = (g \circ f)[f^{-1}(\mathcal{V})]$ is semi-open in Z. This shows that g is a pre-semi-open mapping.

(2) We note that $f(A) = g^{-1}[g(f(A))]$ for every subset A of X because g is injective. Let U be an arbitrary semi-open set in X. Then $(g \circ f)(U)$ is semi-open. Now $f(U)$ is semi-open in Y since g is irresolute by Theorem 3.38. This shows that f is a semi-open mapping.

3.40. THEOREM. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two mappings such that* $g \circ f : X \to Z$ *is irresolute. Then*

(1) If g *is a pre-semi-open injection, then* f *is irresolute.*

(2) If f *is a pre-semi-open surjection, then* g *is irresolute.*

PROOF:

(1) Let $U \in SO(Y)$. Then $g(U) \in SO(Z)$ since g is pre-semi-open. Also $g \circ f$ is irresolute. Therefore, we have $(g \circ f)^{-1}(g(U)) \in SO(X)$.

Since g is an injection, so we have:

$$(g \circ f)^{-1} g(U) = (f^{-1} \circ g^{-1}) g(U) = f^{-1}[g^{-1} g(U)] = f^{-1}(U).$$

Consequently $f^{-1}(U)$ is semi-open in X. This proves that f is irresolute.

(2) Let $\mathcal{V} \in SO(Z)$. Then $(g \circ f)^{-1}(\mathcal{V}) \in SO(X)$ since $g \circ f$ is irresolute. Also f is pre-semi-open, so $f(g \circ f)^{-1}(\mathcal{V})$ is semi-open in Y. Since f is surjective, we note that

$$f \circ (g \circ f)^{-1}(\mathcal{V}) = f \circ (f^{-1} \circ g^{-1})(\mathcal{V}) = (f \circ f^{-1}) \circ g^{-1}(\mathcal{V}) = g^{-1}(\mathcal{V}).$$

It follows that $g^{-1}(\mathcal{V}) \in SO(Y)$. Thus g is an irresolute function.

E. PRE-SEMI-CLOSED FUNCTIONS

In this section, we explore certain new properties and characterizations of pre-semi-closed functions.

3.41. DEFINITION: A function $f : X \to Y$ is pre-semi-closed if and only if the image set $f(A)$ is semi-closed for each semi-closed subset A of X.

3.42. THEOREM. *The composition of two pre-semi-closed mappings is a pre-semi-closed mapping.*

PROOF: The straight forward proof is omitted.

3.43. THEOREM. *A function f is a pre-semi-closed mapping if and only if*

$$scl\ f(A) \subseteq f[scl(A)] \quad \textit{for every subset } A \textit{ of } X.$$

PROOF: NECESSITY. Suppose f is pre-semi-closed and A is an arbitrary subset of X. Then $f[scl(A)]$ is semi-closed in Y. Since $f(A) \subseteq f[scl(A)]$, we obtain $scl\ f(A) \subseteq f[scl(A)]$.

SUFFICIENCY. Suppose F is an srbitrary semi-closed set in X. By hypothesis, we obtain

$$f(F) \subseteq scl[f(F)] \subseteq f[scl(F)] = f(F).$$

Hence $f(F) = scl[f(F)]$. Thus $f(F)$ is semi-closed in Y. It follows that f is semi-closed.

3.44. THEOREM. *Prove that a function $f : X \to Y$ is a pre-semi-closed mapping if and only if Int $cl[f(A)] \subseteq f(scl A)$ for every subset A of X.*

PROOF: NECESSITY. Suppose f is a pre-semi-closed mapping and A is an arbitrary subset of X. Then $f(scl A)$ is semi-closed in Y. So by definition there exists a closed subset F of Y such that Int $F \subseteq f(scl\ A) \subseteq F$. This implies that Int $cl[f(scl\ A)] \subseteq$ Int $F \subseteq f(scl\ A)$. But also Int $cl[f(A)] \subseteq$ Int $cl[f(scl\ A)]$. Hence Int $cl[f(A)] \subseteq f(scl\ A)$.

SUFFICIENCY. Suppose F is an arbitrary semi-closed set in X. Then by hypothesis, we have

$$\text{Int } cl[f(F)] \subseteq f(scl\ F) = f(F).$$

Take $E = cl\ f(F)$. Then E is closed in Y. Also it implies that Int $E \subseteq f(F) \subseteq E$. Hence $f(F)$ is semi-closed in Y. This implies that f is pre-semi-closed.

3.45. THEOREM. *Let $f : X \to Y$ be a pre-semi-closed function, and $B, C \subseteq Y$.*

(1) If U is a semi-open neighbourhood of $f^{-1}(B)$, then there exists a semi-open neighbourhood $\mathcal{V}$ of B such that $f^{-1}(B) \subseteq f^{-1}(\mathcal{V}) \subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint semi-neighbourhoods, so have B and C.

PROOF:

(1) Let $\mathcal{V} = Y - f(X - U)$. Then $\mathcal{V}^c = Y - \mathcal{V} = f(U^c)$. Since f is pre-semi-closed, so $\mathcal{V}$ is semi-open. Since $f^{-1}(B) \subseteq U$, we have

$$\mathcal{V}^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c.$$

Hence, $B \subseteq \mathcal{V}$, and thus $\mathcal{V}$ is a semi-open neighbourhood of B. Further $U^c \subseteq f^{-1}f(U^c) = f^{-1}(\mathcal{V}^c) = [f^{-1}(\mathcal{V})]^c$. This proves that $f^{-1}(\mathcal{V}) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint semi-neighbourhoods M and N, then by (1), we have semi-open neighbourhoods U and V of B and C respectively such that

$$f^{-1}(B) \subseteq f^{-1}(U) \subseteq s\,\text{Int}\,M$$

and

$$f^{-1}(C) \subseteq f^{-1}(V) \subseteq s\,\text{Int}\,N.$$

Since M and N are disjoint, so are $s\,\text{Int}\,M$ and $s\,\text{Int}\,N$, and hence so are $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

3.46. THEOREM. *A surjective mapping $f : X \to Y$ is pre-semi-closed if and only if for each subset B of Y and each semi-open set U in X containing $f^{-1}(B)$, there exists a semi-open set V in Y containing B such that $f^{-1}(V) \subseteq U$.*

PROOF: NECESSITY. This follows from (1) of Theorem 3.45.

SUFFICIENCY. Suppose F is an arbitrary semi-closed set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is semi-open in X. Hence by hypothesis, there

exists a semi-open set V_y containing y such that $f^{-1}(V_y) \subseteq X - F$. This implies that $y \in V_y \subseteq Y - f(F)$. Thus

$$Y - f(F) = \cup\{V_y : y \in Y - f(F)\}.$$

Hence $Y - f(F)$, being a union of semi-open sets is semi-open. Thus its complement $f(F)$ is semi-closed. This shows that f is semi-closed.

3.47. THEOREM. *Let $f : X \to Y$ be a bijection. Then the following are equivalent.*

(a) f is pre-semi-closed.

(b) f is pre-semi-open.

(c) f^{-1} is an irresolute.

PROOF: (a) $\Longrightarrow$ (b) : Let $U \in SO(X)$. Then $X - U$ is semi-closed in X. By (a), $f(X - U)$ is semi-closed in Y. But

$$f(X - U) = f(X) - f(U) = Y - f(U).$$

Thus $f(U)$ is semi-open in Y. This shows that f is pre-semi-open.

(b) $\Longrightarrow$ (c) : Let $A \subseteq X$. Since f is pre-semi-open, by Theorem 3.37,

$$f^{-1}[scl\ f(A)] \subseteq scl\ f^{-1}f(A)$$

i.e., $scl\ f(A) \subseteq f(scl\ A)$. Thus $scl(f^{-1})^{-1}(A) \subseteq (f^{-1})^{-1}(scl\ A)$, for all $A \subseteq X$. Then by Theorem 3.16, it follows that f^{-1} is an irresolute.

(c) $\Longrightarrow$ (a) : Let A be an arbitrary semi-closed set in X. Then $X - A$ is semi-open in X. Since f^{-1} is an irresolute, $(f^{-1})^{-1}(X - A)$ is semi-open in Y. But

$$(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A).$$

Thus $f(A)$ is semi-closed in Y. This shows that f is pre-semi-closed.

CHAPTER IV

D-SPACES

In 1968, N. Levine introduced the concept of a D-space by calling a topological space (X,T) a D-space whenever every nonempty open set is dense in X. In this chapter we study a characterization of D-spaces in terms of semi-open sets as well as mappings on D-spaces.

A. CHARACTERIZATION OF *D*-SPACES

In this section, we give some characterizations of D-spaces.

4.1. THEOREM. *In a topological space* (X,T), *the following are equivalent:*

(i) (X,T) *is a* D*-space.*

(ii) Every pair of nonempty open sets has a nonempty intersection.

(iii) Every open set in X *is connected.*

PROOF: Easy.

4.2. THEOREM. *A topological space* X *is a* D*-space if and only if for each nonempty semi-open set* S *in* X, $scl\ S = X$.

PROOF: NECESSITY. Let S be a nonempty semi-open set in X. Let $x \in X$. Let $T \in SO(X)$ such that $x \in T$. Then there exist nonempty open sets U and V in X such that $U \subseteq T \subseteq \overline{U}$ and $V \subseteq S \subseteq \overline{V}$. By hypothesis, $\overline{U} = X = \overline{V}$. If $S \cap T = \phi$, then $U \cap V = \phi$. Then by Theorem 4.1, X

is not a D-space. But this is a contradiction. Hence $S \cap T \neq \phi$. Thus $x \in scl\, S$, so $X \subseteq scl\, S$. Hence, $X = scl\, S$.

SUFFICIENCY. Suppose that for each nonempty semi-open set S in X, we have $scl\, S = X$. Let U be a nonempty open set in X. Then U is a nonempty semi-open set in X. Hence by hypothesis, $scl\, U = X$. But $scl\, U$ being the intersection of all semi-closed sets containing U, and each closed set being a semi-closed set, it follows immediately that $scl\, U \subseteq cl\, U$. Thus $cl\, U = X$. Therefore, X is a D-space.

4.3. THEOREM. *A topological space X is a D-space if and only if every pair of nonempty semi-open sets has a nonempty intersection.*

PROOF: NECESSITY. Suppose X is a D-space. Let S and T be nonempty semi-open sets in X. Then there exist U and V nonempty open sets in X such that $U \subseteq S \subseteq cl\, U$, $V \subseteq T \subseteq cl\, V$. By Theorem 4.1, $U \cap V = \phi$. Hence, $S \cap T \neq \phi$.

SUFFICIENCY. Let S be a nonempty semi-open set in X. Let $x \in X$. Let $T \in SO(X)$ such that $x \in T$. By hypothesis, $S \cap T \neq \phi$. Hence, $x \in scl\, S$. Thus $X = scl\, S$. So by Theorem 4.2, X is a D-space.

We note that a D-space is never a semi-T_2 space though it can be a semi-T_1 space.

Now in the following, we define S-connected spaces and examine their relation with D-spaces.

4.4. DEFINITION: A space X is said to be S-connected if X cannot be written as the disjoint union of two nonempty semi-open sets.

We observe that an S-connected space is connected but a connected space may not be an S-connected space. For example, R with the usual topology.

4.5. THEOREM. *Every D-space is S-connected and hence connected.*

PROOF: Suppose X is a D-space. If X is not S-connected, then there exist two nonempty semi-open sets S and T such that $X = S \cup T$ and $S \cap T = \phi$. Hence $scl\ S \neq X$. But then by Theorem 4.2, X is not a D-space which is a contradiction. Thus X is S-connected.

We note that a connected space may not be a D-space. For example , R with the usual topology.

4.6. THEOREM. *A topological space X is a D-space if and only if every nonempty semi-open set in X is S-connected.*

PROOF: NECESSITY. Let Y be a nonempty semi-open set in X. If Y is not S-connected, then there exist two nonempty semi-open sets T_1 and T_2 in Y such that $Y = T_1 \cup T_2$ and $T_1 \cap T_2 = \phi$. Then T_1 and T_2 are two disjoint nonempty semi-open sets in X. So by Theorem 4.3, X is not a D-space. But this is a contradiction. Hence Y is S-connected.

SUFFICIENCY. Suppose that X is not a D-space. Then by Theorem 4.1, there exist nonempty open sets U and V in X such that $U \cap V = \phi$. Let $Y = U \cup V$. Then Y is semi-open in X. Also U and V are open in Y and hence semi-open in Y. Thus Y is not S-connected. This contradiction shows that X is a D-space.

4.7. THEOREM. *A space with a dense D-subspace is itself a D-space.*

PROOF: Theorem 5(i) of (LEVINE [1968]).

4.8. THEOREM. *If A is a D-subspace of a space X, then $scl\, A$ is also a D-subspace of X.*

PROOF: Since A is dense in $scl\, A$, the result follows from Theorem 4.7.

4.9. THEOREM. *Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace $X - C$ of X has an open set V. Then $C \cup V$ is a D-subset of X.*

PROOF: Theorem 4.9 of [FATTEH AND SINGH [1983]).

Our next result is a generalization of this Theorem.

4.10. THEOREM. *Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace $X - C$ of X has a semi-open set S. Then $C \cup S$ is a D-subset of X.*

PROOF: Since every nonempty semi-open set contains a nonempty open set, the result follows from Theorem 4.9.

In the following, we give a characterization of D-spaces in terms of regular semi-open sets. We need the following result.

4.11. THEOREM. *A topological space X is a D-space if and only if X has no proper regularly open set.*

PROOF: Theorem 3.2 of (FATTEH AND SINGH [1983]).

4.12. DEFINITION: A subset A of a topological space (X, τ) is regular semi-open if there is a regular open set U such taht

$$U \subseteq A \subseteq cl\, U.$$

4.13. THEOREM. *A topological space X is a D-space if and only if X has no proper regularly semi-open set.*

PROOF: NECESSITY. If X has a proper regularly semi-open set A, then there exists a regularly open set U such that $U \subseteq A \subseteq cl\, U$. Clearly U is also proper. Hence by Theorem 4.11, X is not a D-space. This contradiction shows that X does not have any proper regularly semi-open set.

SUFFICIENCY. Suppose X is not a D-space. Then by Theorem 4.11, there exists a proper regularly open set U. Clearly U is a proper regularly semi-open set which is a contradiction. Thus X is a D-space.

Now we give a characterization of semi-open sets in a D-space. We need the following easy lemma.

4.14. LEMMA. *Let* (X,T) *be a* D*-space. Let* U *and* V *be non-empty open sets in* X. *Then* $cl(U \cap V) = (cl\, U) \cap (cl\, V)$.

PROOF: Easy.

4.15. THEOREM. *Let* (X,T) *be a* D*-space. Let* $A, B \in SO(X)$. *Then* $A \cap B \in SO(X)$.

PROOF: Let $U, V \in T$ such that

$$U \subseteq A \subseteq cl\, U, \quad V \subseteq B \subseteq cl\, V.$$

Then $U \cap V \subseteq A \cap B \subseteq (cl\, U) \cap (cl\, V)$. By applying Lemma 4.14, we have:

$$U \cap V \subseteq A \cap B \subseteq cl(U \cap V).$$

Thus $A \cap B \in SO(X)$.

4.16. THEOREM. *If* (X,T) *is a topological space and* T *is finite, then* (X,T) *is a* D*-space if and only if the intersection of all non-void semi-open subsets of* X *is non-void.*

PROOF: NECESSITY. If possible then suppose that $\cap(SO(X) - \{\phi\}) = \phi$. Then we assert that $\cap(T - \{\phi\}) = \phi$. For that let $x \in X$. Then there exists $S_x \in SO(X) - \{\phi\}$ such that $x \notin S_x$. So there exists $U_x \in T - \{\phi\}$ such that $U_x \subseteq S_x \subseteq cl\ U_x$. Hence $x \notin U_x$. Thus $\cap(T - \{\phi\}) = \phi$. Then it follows easily by Theorem 4.1 that X is not a D-space. By this contradiction, we conclude that $\cap(SO(X) - \{\phi\}) \neq \phi$.

SUFFICIENCY. Suppose that $\cap(SO(X) - \{\phi\}) \neq \phi$. Since $T \subseteq SO(X)$. Hence $\cap(T - \{\phi\}) \neq \phi$. Then by Theorem 4.1, it follows immediately that (X, T) is a D-space.

B. CHARACTERIZATION OF IMAGES AND PRE-IMAGES OF *D*-SPACES

In the present section, we will show that some mappings preserve the D-space structure of their images and pre-images.

4.17. THEOREM. *The semi-continuous image of a D-space is a D-space.*

PROOF: Let $f : X \to Y$ be a surjective semi-continuous function where X is a D-space. If Y is not a D-space, then by Theorem 4.1, there exist nonempty open sets U and V in Y such that $U \cap V = \phi$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint semi-open sets. So by Theorem 4.3, X is not a D-space. But this is a contradiction. Hence Y is a D-space.

4.18. COROLLARY. *The irresolute image of a D-space is a D-space.*

4.19. THEOREM. *Let (X,T) and (Y,T') be topological spaces and suppose that $f:X\to Y$ is a function. Then prove the following.*

(i) If f is one to one and semi-open and if (Y,T') is a D-space , then (X,T) is a D-space.

(ii) If f is one to one and pre-semi open and if (Y,T') is a D-space, then (X,T) is a D-space.

(iii) If f is onto and $SO(X)=\{f^{-1}(U):U\in T'\}$, then (X,T) is a D-space if and only if (Y,T') is a D-space.

(iv) If f is onto and $SO(X)=\{f^{-1}(U)|U\in SO(Y)\}$, then (X,T) is a D-space if and only if (Y,T') is a D-space.

PROOF:

(i) If (X,T) is not a D-space, then by Theorem 2.1, there exist disjoint nonempty open sets A and B in X. Then $f(A)$ and $f(B)$ are disjoint nonempty semi-open sets in Y and by Theorem 4.3, Y is not a D-space. But this is a contradiction to our hypothesis, so (X,T) is a D-space.

(ii) The straight forward proof (similar to (i)) is omitted.

(iii) Necessity follows from Theorem 4.17.

To show the sufficiency, suppose that (X,T) is not a D-space. Then by Theorem 4.3, there exist disjoint nonempty semi-open sets A and B in

X. Then there exist non-empty open sets U and V in Y such that $A = f^{-1}(U)$ and $B = f^{-1}(V)$. Now $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) = A \cap B = \phi$. Since f is onto. So $U \cap V = \phi$. Thus U and V are disjoint non-empty open sets in Y. So by Theorem 4.1, Y is not a D-space. But this is a contradiction. Hence (X,T) is a D-space.

(iv) The necessity follows from Corollary 4.18, To prove sufficiency, suppose (X,T) is not a D-space. Then by Theorem 4.3, there exist disjoint nonempty semi-open sets A and B in X. Then

$$\phi = A \cap B = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$$

for some nonempty semi-open sets U and V in Y. Also U and V are disjoint as f is onto. Then by Theorem 4.3, it follows that Y is not a D-space. This leads us to a contradiction. Hence it is proved that (X,T) is a D-space.

4.20. THEOREM. *Every semi-continuous function from a D-space into a T_2-space is a constant function.*

PROOF: Let $f : X \to Y$ be semi-continuous, where X is a D-space and Y is a T_2-space. Let $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$. Since Y is T_2, so there exist disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$.

Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint nonempty semi-open sets in X. Hence by Theorem 4.3, X is not a D-space. But this is a contradiction. Thus f is a constant function.

4.21. THEOREM. *Prove that every irresolute function from a D-space into semi-T_2 space is constant.*

PROOF: Let $f : X \to Y$ be an irresolute function, where X is a D-space and Y is a semi-T_2 space. Let $x, y \in X$ such that $f(x) \neq f(y)$. Since Y is semi-T_2, there exist disjoint semi-open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is an irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty semi-open sets in X. Hence by Theorem 4.3, X is not a D-space. This gives a contradiction. Hence f is a constant function.

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